13. Kompleksni brojevi

13. 1. Algebarski oblik kompleksnog broja

- Skup kompleksnih brojeva je skup
  $C = \{ z = x + iy \mid i^2 = -1,\ x, y \in R \}$

- Napomena: Pogrešno je koristiti $\sqrt{-1} = i$, ispravno je $\sqrt{-1} \in \{-i, i\} \Leftrightarrow$
$\Leftrightarrow \sqrt{-1} = \pm i$ (samo za kompleksni koren)
Znači za realni koren je $\sqrt{4} = 2$,
a za kompleksni koren je $\sqrt{4} \in \{-2, 2\}$ tj. $\sqrt{4} = \pm 2$.

- Izraz $x + iy$ nazivamo algebarski oblik kompleksnog broja. Realan broj x nazivamo realan deo kompleksnog broja z, a realan broj y imaginaran deo i koristimo oznake $x = \operatorname{Re} z$, $y = \operatorname{Im} z$

Ako je $\operatorname{Im} z = 0$, onda je z realan broj
Primer: $3, -5, 0, 1 \ldots$
Ako je $\operatorname{Re} z = 0$, onda je z čisto imaginaran broj
Primer: $3i, -2i \ldots$

* Osnovne operacije
Jednakost kompleksnih brojeva se definiše na sledeći način:
$z_1 = z_2 \Leftrightarrow \operatorname{Re} z_1 = \operatorname{Re} z_2 \ \wedge\ \operatorname{Im} z_1 = \operatorname{Im} z_2$

Sabiranje, oduzimanje, množenje i deljenje se izvode na sledeći način:

$$z_1 \pm z_2 = x_1 + iy_1 \pm (x_2 \pm iy_2) = x_1 \pm x_2 + i(y_1 \pm y_2)$$

$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2} = \frac{x_1x_2 + y_1y_2 + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}, \quad z_2 \neq 0$$

Konjugovano kompleksan broj za $z = x + iy$ je $\bar{z} = x - iy$

Za konjugovanje kompleksnih brojeva važe sledeće osobine:

1. $\bar{\bar{z}} = z$
2. $Re\, z = \frac{z + \bar{z}}{2}$ i $Im\, z = \frac{z - \bar{z}}{2i}$
3. $z \cdot \bar{z} = x^2 + y^2$
4. $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$
5. $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$
6. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \; z_2 \neq 0$

Kvadratni koren kompleksnog broja može se izračunati u algebarskom obliku. Ako je $\sqrt{z} = a + ib$, kvadriranjem dobijamo

$x + iy = a^2 + 2abi + i^2b^2 = a^2 - b^2 + 2abi.$

Upoređivanjem realnih i imaginarnih delova dobijamo da je:

$x = a^2 - b^2$ i $y = 2ab$.

Rešavanjem ovog sistema dobijamo realni deo a i imaginarni deo b traženog korena.

1. $z_1 = 3+i$

$z_2 = 1-2i$

a) $3z_1 - 2z_2 = 3(3+i) - 2(1-2i) = 9+3i-2+4i = 7+7i = 7(1+i)$

b) $\frac{z_1}{z_2} = \frac{3+i}{1-2i} \cdot \frac{1+2i}{1+2i} = \frac{3+6i+i-2}{1+4} = \frac{1+7i}{5} = \frac{1}{5} + \frac{7i}{5}$

2. a) $z_1 = 3+i$, $\overline{z_1} = 3-i$ b) $z_2 = 1-2i$, $\overline{z_2} = 1+2i$ c) $z_3 = 5$, $\overline{z_3} = 5$ d) $z_4 = -i$, $\overline{z_4} = i$

3.

$$Re\left(\frac{\overbrace{(1+i)z+2-2i}^{z}}{3+2i}\right) = Im\left(\frac{(1+i)z-2+2i}{3+2i}\right) = 1$$

$$\frac{(1+i)z+2-2i}{3+2i} = 1+i$$

$$(1+i)z+2-2i = (3+2i)(1+i)$$

$$(1+i)z = (3+2i)(1+i)+2i-2$$

$$z = \frac{(3+2i)(1+i)+2i-2}{1+i} = \frac{3+3i+2i-2+2i-2}{1+i} =$$

$$= \frac{7i-1}{1+i} \cdot \frac{1-i}{1-i} = \frac{7i+7-1+i}{1+1} = \frac{8i+6}{2} = 3+4i$$

4. $\sqrt{3+4i} = a+ib$

$a^2-b^2 = 3 \quad \wedge \quad 2ab = 4$

$a^2 - \frac{4}{a^2} = 3 \quad | \cdot a^2 \qquad b = \frac{2}{a}$

$a^4 - 3a^2 - 4 = 0$

$a^2 = t$

$t^2 - 3t - 4 = 0$

$t_{1,2} = \frac{3 \pm \sqrt{9+16}}{2} \to t_1 = 4, \quad \cancel{t_2 = -1}$

$a^2 = 4$

$a_1 = 2 \quad a_2 = -2 \quad b_1 = 1 \quad b_2 = -1$

$z_1 = 2+i$

$z_2 = -2-i$

5. $z^2 + (1-2i)z - \left(4i - \frac{1}{2}\right) = 0$

$Re(z_1 - z_2)$

$Im(z_1 - z_2)$

$$z_{1,2} = \frac{2i-1 \pm \sqrt{1-4i-4+16i-2}}{2} = \frac{2i-1 \pm \sqrt{12i-5}}{2} =$$

$\sqrt{12i-5} = a+ib$

$$= \frac{2i-1 \pm (2+3i)}{2}$$

$a^2 - b^2 = -5 \qquad 2ab = 12$

$a^2 - \frac{36}{a^2} = -5 \;/\cdot a^2 \qquad b = \frac{6}{a}$

$$z_1 = \frac{2i-1+2+3i}{2} = \frac{5i+1}{2} = \frac{1}{2} + \frac{5i}{2}$$

$a^4 - 36 + 5a^2 = 0$

$$z_2 = \frac{2i-1-2-3i}{2} = \frac{-i-3}{2} = -\frac{3}{2} - \frac{i}{2}$$

$a^2 = t$

$t^2 + 5t - 36 = 0$ 13

$t_{1,2} = \frac{-5 \pm \sqrt{25+144}}{2}$ $\to$ ~~$t_1 = -9$~~ $\to t_2 = 4$

$$Re(z_1 - z_2) = \frac{1}{2} + \frac{3}{2} = 2$$

$$Im(z_1 - z_2) = \frac{5}{2} + \frac{1}{2} = 3$$

$a^2 = 4$

$a_1 = 2 \qquad a_2 = -2$

$b_1 = 3 \qquad b_2 = -3$

$2+3i$

$-2-3i$

6. $\sqrt{\frac{1}{2^{10}}\left(\frac{5-14i}{4-i}-i\right)^7+1+2i}$

$$\frac{1}{2^{10}}\left(\frac{5-14i}{4-i}+i\right)^7=\frac{1}{2^{10}}\left(\frac{(5-14i)(4+i)}{17}+i\right)^7=\frac{1}{2^{10}}\cdot\left(\frac{20+5i-56i+14}{17}+i\right)^7=$$

$$=\frac{1}{2^{10}}\left(\frac{34-51i}{17}+i\right)^7=\frac{1}{2^{10}}\left(\frac{34-34i}{17}\right)^7=\frac{1}{2^{10}}\left(2\cdot(1-i)\right)^7=\frac{1}{2^{10}}\cdot 2^7\cdot(1-i)^7=$$

$$=\frac{1}{8}\cdot(1-i)^7=\frac{1}{8}(1-i)^2(1-i)^2(1-i)^2(1-i)=\frac{1}{8}(\cancel{1}-2i-\cancel{1})(1-i)^2(1-i)^2(1-i)=$$

$$=\frac{1}{8}(-2i)(-2i)(-2i)(1-i)=\frac{1}{\cancel{8}}\cancel{8}i(1-i)=i+1$$

$\sqrt{3i+2}=x+yi$

$3i+2=x^2+2xyi-y^2$

$x^2-y^2=2 \qquad 2xy=3$

$x=\frac{3}{2y}$

$\frac{9}{4y^2}-y^2=2 \quad |\cdot y^2$

$\frac{9}{4}-y^4-2y^2=0 \quad |\cdot 4$

$9-4y^4-8y^2=0 \qquad y^2=t$

$4t^2+8t-9=0$

ne može – zbog $y^2\geq 0$

$$t_{1,2}=\frac{-8\pm\sqrt{64+144}}{8}=\frac{-8\pm4\sqrt{13}}{8}=\frac{4(-2\pm\sqrt{13})}{\cancel{8}_2}=\frac{-2\pm\sqrt{13}}{2}=\frac{-2+\sqrt{13}}{2}=\frac{\sqrt{13}}{2}-1$$

$y^2=t$

$y_{1,2}=\pm\sqrt{\frac{\sqrt{13}}{2}-1} \qquad x_{1,2}=\pm\frac{3}{2\sqrt{\frac{\sqrt{13}}{2}-1}}$

$$z_1 = \frac{3}{2\sqrt{\frac{\sqrt{13}}{2}-1}} + i\sqrt{\frac{\sqrt{13}}{2}-1}$$

$$z_2 = -z_1$$

7. $\sqrt{-7+24i}$ $\quad \sqrt[4]{-7+24i}$

$x_2 - y_2 = -7 \qquad 2xy = 24$

$y = \frac{12}{x}$

$x^2 - \frac{144}{x^2} + 7 = 0$

$x^4 - 144 + 7x^2 = 0 \qquad x^2 = t$

$t^2 + 7t - 144 = 0$

$$t_{1,2} = \frac{-7 \pm \sqrt{\overset{25}{49+576}}}{2} \rightarrow t = 9, \quad \cancel{t_2 = -16}$$

$x^2 = 9$

$x_1 = 3 \qquad x_2 = -3$

$y_1 = 4 \qquad y_2 = -4$

$\sqrt{3+4i}$

$x^2 - y^2 = 3 \qquad 2xy = 4$

$x^2 - \frac{4}{x^2} - 3 = 0 \qquad y = \frac{2}{x}$

$x^4 - 3x^2 - 4 = 0 \qquad x^2 = t$

$t^2 - 3t - 4 = 0$

$$t_{1,2} = \frac{3 \pm \sqrt{\overset{5}{9+16}}}{2} \rightarrow t_1 = 4, \quad \cancel{t_2 = -1}$$

$x^2 = 4$

$x_1 = 2 \qquad x_2 = -2$

$y_1 = 1 \qquad y_2 = -1$

$z_1 = 2 + i$

$z_2 = -2 - i$

$\sqrt{-3-4i}$

$x^2 - y^2 = -3$ $\qquad$ $2xy = -4$

$y = -\frac{2}{x}$

$x^2 - \frac{4}{x^2} + 3 = 0$

$x^4 + 3x^2 - 4 = 0$ $\qquad$ $x^2 = t$

$t^2 + 3t - 4 = 0$

$t_{1,2} = \frac{-3 \pm \overset{5}{\sqrt{9+16}}}{2}$ $\quad$ $t_1 = 1$ $\quad$ $\cancel{t_1 = -4}$

$x^2 = 1$

$x_1 = 1$ $\qquad$ $x_2 = -1$

$y_1 = -2$ $\qquad$ $y_2 = 2$

$z_3 = 1 - 2i$

$z_4 = -1 + 2i$

13. 2. Geometrijska interpretacija kompleksnog broja

- Kompleksne brojeve je moguće predstaviti u kompleksnoj ravni tj u xy-koordinatnom sistemu. Svakom kompleksnom broju $z = x + iy$ odgovara jedna i samo jedna tačka $M(x, y)$.

- Moduo kompleksnog broja $|z| = \sqrt{x^2+y^2}$ predstavlja rastojanje tačke M od koordinatnog početka.

- Argument kompleksnog broja $z = x+iy$ koji označavamo sa $\arg z$ je merni broj orijentisanog konveksnog ugla koji vektor $\overrightarrow{OM}$ zaklapa sa pozitivnim smerom realne ose.
Odavde sledi da je $\arg z \in (-\pi, \pi]$
Za $z=0$ argument se ne definiše.

- Kako je konjugovana vrednost $\bar{z}$ simetrična broju z u odnosu na realnu osu, to je $|\bar{z}| = |z| \wedge \arg \bar{z} = -\arg z$

8. a) $z_1 = 2-2i$

$|z| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$\operatorname{tg}\varphi = \frac{y}{x} = \frac{-2}{2} = -1, \quad x>0, \; y<0$

$\arg z = \varphi = -\frac{\pi}{4} + 2k\pi, \quad k = 0, \pm 1, \dots,$

b) $z_2 = 3+0i$

$|z_2| = \sqrt{9} = 3$

$\operatorname{tg}\varphi = \frac{y}{x} = \frac{0}{3} = 0$

$\arg z_2 = \varphi = 0 + 2k\pi$

c) $z_3 = -1+0i$

$|z_3| = \sqrt{1} = 1$

$\operatorname{tg}\varphi = \frac{y}{x} = \frac{0}{-1} = 0$

$\arg z_3 = \varphi = \pi + 2k\pi$

d) $z_4 = 2i$

$|z_4| = \sqrt{4} = 2$

$tg\varphi = \frac{y}{x} = \frac{2}{0} = \infty$

$\arg z_4 = \varphi = \frac{\pi}{2} + 2k\pi$

e) $z_5 = -2i$

$|z_5| = \sqrt{4} = 2$

$tg\varphi = \frac{y}{x} = \frac{-2}{0} = \infty$

$\arg z_5 = \varphi = -\frac{\pi}{2} + 2k\pi$

9. $|z| - z = 1 + 2i$

$\sqrt{x^2+y^2} - x - iy = 1 + 2i$

$\sqrt{x^2+y^2} - x = 1 \qquad y = -2$

$\sqrt{x^2+4} - x = 1$

$\sqrt{x^2+4} = x + 1$

$\cancel{x^2} + 4 = \cancel{x^2} + 2x + 1$

$2x = 3$

$x = \frac{3}{2}$

$z = \frac{3}{2} - 2i$

10. $|z - i| = \operatorname{Im} z \qquad \operatorname{Re} z = \operatorname{Im} z$

$|z - i| = |x + iy - i| = |x + i(y-1)| = \sqrt{x^2 + (y-1)^2} \qquad x = 1$

$x^2 + (y-1)^2 = y^2 \qquad x = y \qquad y = 1$

$\cancel{x^2} + x^2 - 2x + 1 = \cancel{x^2} \qquad z = 1 + i$

$(x-1)^2 = 0$

13.3 Trigonometrijski i eksponencijalni oblik

Ako kompleksni broj $z = x + iy$ predstavimo u kompleksnoj ravni i označimo

$|z| = r$, $\sin\varphi = \frac{y}{r}$, $\cos\varphi = \frac{x}{r}$

vidimo da je $y = r\sin\varphi$, $x = r\cos\varphi$, tj.

$z = x + iy = r\cos\varphi + ir\sin\varphi = r(\cos\varphi + i\sin\varphi)$

Jednakost dva kompleksna broja predstavljena u trigonometrijskom obliku definiše se na sledeći način:

$z_1 = z_2 \iff |z_1| = |z_2| \wedge \arg z_1 = \arg z_2$

Operacije konjugovanja, množenja, deljenja i stepenovanja se izvode na sledeći način:

$\overline{z} = r(\cos(-\varphi) + i\sin(-\varphi))$

Ako je $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$, $z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$, onda je

$z_1 \cdot z_2 = r_1 r_2 (\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2))$

$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2))$

$z^n = r^n(\cos n\varphi + i\sin n\varphi)$, $n \in \mathbb{Z}$

Ako uvedemo oznaku $\cos\varphi + i\sin\varphi = e^{\varphi i}$, onda z možemo predstaviti u obliku $z = re^{\varphi i}$, što zovemo eksponencijalni oblik kompleksnog broja.

Opercije u eksponencijalnom obliku se izvode kao operacije sa običnim stepenima:

$z_1 \cdot z_2 = r_1 e^{\varphi_1 i} \cdot r_2 e^{\varphi_2 i} = r_1 r_2 \cdot e^{i(\varphi_1 + \varphi_2)}$

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}e^{i(\varphi_1-\varphi_2)}$$

$$\overline{z}=re^{-\varphi i}$$

$$z^n=\left(re^{\varphi i}\right)^n=r^n e^{in\varphi},\quad n\in\mathbb{Z}$$

- Posmatrajmo jednačinu $z^n=a$ gde je $a=r(\cos\varphi+i\sin\varphi)$. Njeno rešenje je kompleksan broj $z=\sqrt[n]{a}$, koji ima n različitih vrednosti:

$$z_k=\sqrt[n]{r}\left(\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right),\quad k=0,1,\dots,n-1$$

Ova rešenja leže na kružnici poluprečnika $\sqrt[n]{r}$ i čine temena pravilnog n-tougla čije jedno teme (z_0) ima argument $\frac{\varphi}{n}$, a za svako sledeće teme argument se povećava za $\frac{2\pi}{n}$.

1. $z_1=1+\sqrt{3}i$
   $z_2=1+i$

$|z_1|=\sqrt{1+3}=2$ $\qquad$ $|z_2|=\sqrt{1+1}=\sqrt{2}$

$\operatorname{tg}\varphi=\frac{y}{x}=\sqrt{3}$ $\qquad$ $\operatorname{tg}\varphi=\frac{y}{x}=1$

$\arg z_1=\frac{\pi}{3}$ $\qquad$ $\arg z_2=\frac{\pi}{4}$

$z_1=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ $\qquad$ $z_2=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$z_1=2e^{\frac{\pi}{3}i}$ $\qquad$ $z_2=\sqrt{2}e^{i\frac{\pi}{4}}$

$$z_1\cdot z_2=2\sqrt{2}\left(\cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\right)=2\sqrt{2}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)$$

$$\frac{z_1}{z_2}=\frac{2}{\sqrt{2}}\left(\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\right)=\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$$

$z_1 \cdot z_2 = 2e^{\frac{\pi}{3}i} \cdot \sqrt{2}e^{\frac{\pi}{4}i} = 2\sqrt{2}\,e^{i\frac{7\pi}{12}}$

$\frac{z_1}{z_2} = \frac{2e^{\frac{\pi}{3}i}}{\sqrt{2}e^{\frac{\pi}{4}i}} = \sqrt{2}e^{\frac{\pi}{12}i}$

2. $z = 1+i$

$|z| = \sqrt{2}$

$tg\varphi = 1$

$\arg z = \frac{\pi}{4}$

$(i+1)^{97} = \left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^{97} = 2^{48}\sqrt{2}\left(\cos\frac{97\pi}{4} + i\sin\frac{97\pi}{4}\right) =$

$= 2^{48}\sqrt{2}\left(\cos\left(24\pi + \frac{\pi}{4}\right) + i\sin\left(24\pi + \frac{\pi}{4}\right)\right) =$

$= 2^{48}\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2^{48}\sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) =$

$= 2^{48}\cancel{\sqrt{2}}\,\frac{\cancel{\sqrt{2}}}{\cancel{2}}(1+i) = 2^{48}(1+i) = 2^{48} + 2^{48}i$

3. $z_1 = 1 - i$

$z_2 = -\sqrt{3} + i$

$z_3 = 1 + \sqrt{3}i$

$|z_1| = \sqrt{2}$

$tg\varphi = \frac{y}{x} = \frac{-1}{1} = -1$

$\arg\varphi = -\frac{\pi}{4}$

$|z_2| = \sqrt{3+1} = 2$

$tg\varphi = \frac{y}{x} = \frac{1}{\sqrt{3}}$

$\arg\varphi = \frac{5\pi}{6}$

$|z_3| = \sqrt{4} = 2$

$tg\varphi = \frac{y}{x} = \sqrt{3}$

$\arg\varphi = \frac{\pi}{3}$

$z_1 = \sqrt{2}e^{-\frac{\pi}{4}i}$

$z_2 = 2e^{\frac{5\pi}{6}i}$

$z_3 = 2e^{\frac{\pi}{3}i}$

$\frac{z_1^{10}}{z_2^{9}} \cdot z_3^5 = \frac{2^5 \cdot e^{-\frac{5\pi}{2}i} \cdot 2^5 \cdot e^{\frac{5\pi}{3}i}}{2^9\, e^{\frac{15\pi}{2}i}} = 2 \cdot e^{-\frac{50\pi}{6}i} = 2 \cdot e^{\left(-8\pi - \frac{2\pi}{6}\right)i} = 2 \cdot e^{-\frac{2\pi}{6}i}$

$\frac{z_1^{10}}{z_2^{9}} z_3^5 = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}i$

14. $z=\sqrt[6]{-1}$ $\qquad z^6+1=0$

$z^6=-1$ $\qquad -1=1\left(\cos(\pi+2k\pi)+i\sin(\pi+2k\pi)\right)$

$|z^6|=\sqrt{1}=1$

$tg\,\varphi=\frac{y}{x}=\frac{0}{-1}=0$ $\qquad z=\sqrt[6]{1}\left(\cos\frac{\pi+2k\pi}{6}+i\sin\frac{\pi+2k\pi}{6}\right),\ k=0,1,\dots,5.$

$\arg\varphi=\pi+2k\pi$

$z_0=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}+i\frac{1}{2}$

$z_1=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=0+i=i$

$z_2=\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$

$z_3=\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}=-\frac{\sqrt{3}}{2}-\frac{1}{2}i=-z_0$

$z_4=-z_1=-i$

$z_5=-z_2=\frac{\sqrt{3}}{2}-\frac{1}{2}i$

15. $\sqrt[4]{-8+8\sqrt{3}i}$

$z=-8+8\sqrt{3}i$ $\qquad tg\,\varphi=\frac{y}{x}=\frac{8\sqrt{3}}{-8}=-\sqrt{3}$

$|z|=\sqrt{64+192}$ $\qquad \arg z=\frac{2\pi}{3}$

$|z|=16$

$\sqrt[4]{z}=\sqrt[4]{16}\left(\cos\frac{\frac{2\pi}{3}+2k\pi}{4}+i\sin\frac{\frac{2\pi}{3}+2k\pi}{4}\right),\ k=0,1,2,3.$

$z_0=2\cdot\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=2\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=\sqrt{3}+i$

$z_1=2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=-1+\sqrt{3}i$

$z_2=-z_0=-\sqrt{3}-i$

$z_3=-z_1=1-\sqrt{3}i$

13.4. Zadaci za vežbu

1. a) $\frac{1}{i}\cdot\frac{i}{i}=\frac{i}{-1}=-i$

b) $\frac{1-i}{1+i}\cdot\frac{1-i}{1-i}=\frac{(1-i)^2}{2}=\frac{\cancel{1}-2i-\cancel{1}}{2}=-i$

c) $\frac{2}{1-3i}\cdot\frac{1+3i}{1+3i}=\frac{2+6i}{10}=\frac{1+3i}{5}$

d) $\frac{3i}{i-2}\cdot\frac{i+2}{i+2}=\frac{-3+6i}{-5}=\frac{-(3-6i)}{-5}=\frac{3-6i}{5}$

e) $\frac{1+i}{2-i}(3+2i)=\frac{3+2i+3i-2}{2-i}=\frac{5i+1}{2-i}\cdot\frac{2+i}{2+i}=\frac{10i-5+2+i}{5}=$

$=\frac{11i-3}{5}$

f) $\frac{1-3i}{i+1}-\frac{i}{i+2}=\frac{\cancel{i}+2+3-6i+1-\cancel{i}}{(i+1)(i+2)}=\frac{-6i+6}{-1+2i+i+2}=\frac{6-6i}{3i+1}\cdot\frac{3i-1}{3i-1}=$

$=\frac{18i-6+18+6i}{-10}=\frac{24i+12}{-10}=\frac{12i+6}{-5}$

2. $z=1+i$, $\bar{z}=1-i$

$\frac{z-\bar{z}}{1+z\bar{z}}=\frac{\cancel{1}+i-\cancel{1}+i}{1+1+1}=\frac{2i}{3}$

3. $z_1=i$, $z_2=\frac{1+i}{\sqrt{2}}=$

$\frac{z_1+z_2}{1+z_1z_2}=\frac{\frac{\sqrt{2}i+1+i}{\sqrt{2}}}{1+\frac{i-1}{\sqrt{2}}}=\frac{\frac{i(\sqrt{2}+1)+1}{\cancel{\sqrt{2}}}}{\frac{\sqrt{2}+i-1}{\cancel{\sqrt{2}}}}=\frac{i(\sqrt{2}+1)+1}{(\sqrt{2}-1)+i}\cdot\frac{(\sqrt{2}-1)-i}{(\sqrt{2}-1)-i}=$

$\frac{\cancel{i}+\sqrt{2}+\cancel{1}+\sqrt{2}-\cancel{1}-\cancel{i}}{4-2\sqrt{2}}=\frac{\cancel{2}\sqrt{2}}{\cancel{2}(2-\sqrt{2})}=\frac{\sqrt{2}}{2-\sqrt{2}}$

4. $|z|+z-4=\frac{8i+2}{1+i}$

$$\sqrt{x^2+y^2}+x+iy-4=\frac{8i+2}{1+i}$$

$$\sqrt{x^2+y^2}+x+iy=\frac{8i+2+4+4i}{1+i}$$

$$\sqrt{x^2+y^2}+x+iy=\frac{12i+6}{1+i}\cdot\frac{1-i}{1-i}$$

$$\sqrt{x^2+y^2}+x+iy=\frac{12i+12+6-6i}{2}=\frac{6i+18}{2}=\frac{2(3i+9)}{2}=3i+9$$

$\sqrt{x^2+y^2}+x+iy=3i+9$

$\sqrt{x^2+y^2}+x=9 \qquad iy=3i$

$\sqrt{x^2+9}+x=9 \qquad y=3$

$\cancel{x^2}+9=81-18x+\cancel{x^2}$

$18x=72$

$x=4$

$z=4+3i$

5. $Re\left(\frac{(2+i)z-2-4i}{1+i}\right)=Im\left(\frac{(2+i)z-2-4i}{1+i}\right)=1$

$$\frac{(2+i)z-2-4i}{1+i}=1+i$$

$(2+i)z-2-4i=\cancel{1}+2i-\cancel{1}$

$(2+i)z=6i+2$

$$z=\frac{6i+2}{2+i}\cdot\frac{2-i}{2-i}=\frac{12i+6+4-2i}{5}=\frac{10i+10}{5}=2i+2$$

6. a) $z = i = 0 + i$

$|z| = \sqrt{1} = 1$

$tg\varphi = \frac{y}{x} = \frac{1}{0} = \infty$

$\arg z = \frac{\pi}{2}$

$\sqrt{z} = \pm 1 \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) =$

$= \pm\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \pm\left(\frac{1+i}{\sqrt{2}}\right)$

b) $z = -64 + 0i$

$|z| = \sqrt{64^2} = 64$

$tg\varphi = \frac{y}{x} = \frac{0}{-64} = 0$

$\arg z = \pi$

$\sqrt{z} = \pm\sqrt{64}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$\sqrt{z} = \pm 8 \; (0 + i)$

$\sqrt{z} = \pm 8i$

c) $z = 3 - 4i$

$\sqrt{z} = x + iy$

$3 - 4i = x^2 + 2xyi - y^2$

$x^2 - y^2 = 3 \quad 2xy = -4$

$^2 - \frac{4}{x^2} = 3 \quad | \cdot x^2 \quad y = \frac{-2}{x}$

$^4 - 3x^2 - 4 = 0$

$^2_{1,2} = \frac{3 \pm \sqrt{9+16}}{2} \to 4$

$\to \cancel{-1}$

$^2 = 4$

$= 2 \quad x_2 = -2$

$= -1 \quad y_2 = 1$

$= \pm(2 - i)$

d) $z = -7 + 24i$

$\sqrt{-7+24i} = x + iy$

$-7 + 24i = x^2 + 2xyi - y^2$

$x^2 - y^2 = -7 \qquad 2xy = 24$

$x^2 - \frac{144}{x^2} + 7 = 0 \quad | \cdot x^2 \qquad y = \frac{12}{x}$

$x^4 + 7x^2 - 144 = 0$

$x^2_{1,2} = \frac{-7 \pm \sqrt{49+576}}{2}$ $\overset{25}{}$ $\nearrow 9$, $\searrow \cancel{-16}$

$x^2 = 9$

$x_1 = 3 \qquad x_2 = -3$

$y_1 = 4 \qquad y_2 = -4$

$z = \pm(3+4i)$

7. a) $z = -i = 0 - i$

$|z| = \sqrt{1} = 1$

$tg\varphi = \frac{y}{x} = \frac{-1}{0} = \infty$

$\arg z = -\frac{\pi}{2}$

b) $z = 1 - i$

$|z| = \sqrt{2}$

$tg\varphi = \frac{y}{x} = \frac{-1}{1} = -1$

$\arg z = -\frac{\pi}{4}$

c) $z = \sqrt{3} - i \qquad \arg z = -\frac{\pi}{6}$

$|z| = \sqrt{3+1} = 2$

$tg\varphi = \frac{y}{x} = -\frac{1}{\sqrt{3}}$

d) $z = 1 + i\sqrt{3}$

$|z| = \sqrt{1+3} = 2$

$tg\varphi = \frac{y}{x} = \sqrt{3}$

$\arg z = \frac{\pi}{3}$

8. $\sqrt{\frac{7}{2}\left(\frac{5-i}{2-3i}\right)^7 - 23 + 40i}$

$\left(\frac{5-i}{2-3i} \cdot \frac{2+3i}{2+3i}\right)^7 = \left(\frac{10+15i-2i+3}{13}\right)^7 = \left(\frac{13i+13}{13}\right)^7 = (i+1)^7$

$z = i+1 = \sqrt{2}\, e^{\frac{\pi}{4}i}$

$|z| = \sqrt{2}$

$tg\varphi = \frac{y}{x} = 1$

$\arg z = \frac{\pi}{4}$

$z^7 = (\sqrt{2})^7 \cdot e^{\frac{7\pi}{4}i}$

$= \sqrt{2} \cdot 2^3 \cdot \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$

$= \cancel{\sqrt{2}} \cdot 2^3 \frac{\cancel{\sqrt{2}}}{\cancel{2}}(1-i)$

$= 2^3(1-i)$

$\sqrt{\frac{7}{\cancel{2}}\overset{4}{\cancel{8}} \cdot (1-i) - 23 + 40i} =$

$= \sqrt{28 - 28i - 23 + 40i} =$

$= \sqrt{12i + 5}$

$\sqrt{12i+5} = x + iy$

$12i + 5 = x^2 + 2xyi - y^2$

$x^2 - y^2 = 5 \qquad 2xyi = 12i$

$x^2 - \frac{36}{x^2} - 5 = 0 \qquad 2xy = 12$

$x^4 - 5x^2 - 36 = 0 \qquad y = \frac{6}{x}$

$x^2_{1,2} = \frac{5 \pm \sqrt{25+144}}{2}$, $\sqrt{25+144} = 13$, $x^2_{1,2} \to 9$, $x^2_{1,2} \to \cancel{-4}$

$x^2 = 9$

$x_1 = 3 \qquad x_2 = -3$

$y_1 = 2 \qquad y_2 = -2$

$z = \pm(3 + 2i)$

9. $\sqrt{56\left(\frac{1+2i}{3+i}\right)^6+\frac{45}{4}}$

$\left(\frac{1+2i}{3+i}\cdot\frac{3-i}{3-i}\right)^6=\left(\frac{3-i+6i+2}{10}\right)^6=\left(\frac{5i+5}{10}\right)^6=\left(\frac{i+1}{2}\right)^6=\left(\frac{1}{2}+\frac{i}{2}\right)^6$

$z=\frac{1}{2}+\frac{i}{2}=\left(\frac{1}{\sqrt{2}}\right)e^{\frac{\pi}{4}i}$

$|z|=\sqrt{\frac{1}{4}+\frac{1}{4}}=\sqrt{\frac{2}{4}}=\frac{\sqrt{2}}{2}$

$\operatorname{tg}\varphi=\frac{y}{x}=1$

$\arg z=\frac{\pi}{4}$

$z^6=\frac{1}{2^3}\cdot e^{\frac{3\pi}{2}i}$

$=\frac{1}{8}\cdot(0-i)$

$=-\frac{1}{8}i$

$\sqrt{\cancel{56}^{\,7}\left(\frac{-1}{\cancel{8}}\right)i+\frac{45}{4}}=\sqrt{-7i+\frac{45}{4}}$

$\sqrt{-7i+\frac{45}{4}}=x+iy$

$-7i+\frac{45}{4}=x^2+2xyi-y^2$

$x^2-y^2=\frac{45}{4}$ $\quad 2xy=-7$

$4x^2-4y^2=45$ $\quad y=\frac{-7}{2x}$

$4x^2-\cancel{4}\cdot\frac{49}{\cancel{4}x^2}=45$

$4x^4-49-45x^2=0$

$x^2_{1,2}=\frac{45\pm\overset{53}{\sqrt{2025+784}}}{8}\ \begin{cases}\to\frac{49}{4}\\ \to\cancel{-1}\end{cases}$

$x^2=\frac{49}{4}$

$x_1=\frac{7}{2}$ $\quad x_2=-\frac{7}{2}$

$y_1=-1$ $\quad y_2=1$

$z=\pm\left(\frac{7}{2}-i\right)$

10. $\mathrm{Im}\left(\frac{1+z}{4+4i}+iz\right) = -\mathrm{Re}\left(\frac{1+z}{4+4i}+iz\right) = 3$

$$\frac{1+z}{4+4i} + iz = -3+3i$$

$$\frac{1+z+4iz-4z}{4+4i} = -3+3i$$

$$-3z+4iz+1 = -12+\cancel{12i}-\cancel{12i}-12$$

$$-3z+4iz = -25$$

$$z(-3+4i) = -25$$

$$z = \frac{-25}{-(3-4i)} = \frac{25}{3-4i}\cdot\frac{3+4i}{3+4i} = \frac{75+100i}{25} = 3+4i$$

$$z = 3+4i$$

$$(\sqrt{z})^2 = 3+4i$$

$$x^2+2xyi-y^2 = 3+4i$$

$$x^2-y^2=3 \qquad 2xy=4$$

$$x^2-\frac{4}{x^2}-3=0 \qquad y=\frac{2}{x}$$

$$x^4-3x^2-4=0$$

$$x^2_{1,2} = \frac{3\pm\sqrt{9+16}^{\,5}}{2} \nearrow 4, \searrow \cancel{-1}$$

$$x^2=4$$

$$x_1=2 \qquad x_2=-2$$

$$y_1=1 \qquad y_2=-1$$

$$z=\pm(2+i)$$

11. $\dfrac{(1-i)^5-1}{(1+i)^5+1}$

$z_1 = 1-i = \sqrt{2}\, e^{-\frac{\pi}{4}i}$ $\qquad z_1^5 = \sqrt{2}\cdot 4\, e^{-\frac{5\pi}{4}i}$

$|z_1| = \sqrt{2}$ $\qquad = 4\sqrt{2}\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)$

$tg\varphi = \frac{y}{x} = -1$ $\qquad = 4\cancel{\sqrt{2}}\frac{\cancel{\sqrt{2}}}{\cancel{2}}(i-1)$

$\arg z = -\frac{\pi}{4}$ $\qquad = 4i-4$

$z_2 = 1+i = \sqrt{2}\, e^{\frac{\pi}{4}i}$ $\qquad z_2^5 = 4\sqrt{2}\, e^{\frac{5\pi}{4}i}$

$|z_2| = \sqrt{2}$ $\qquad = 4\sqrt{2}\left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)$

$tg\varphi = \frac{y}{x} = 1$ $\qquad = 4\cancel{\sqrt{2}}\cdot\frac{\cancel{\sqrt{2}}}{\cancel{2}}(-1-i)$

$\arg z_2 = \frac{\pi}{4}$ $\qquad = -4-4i$

$$\frac{(1-i)^5-1}{(1+i)^5+1} = \frac{4i-4-1}{-4-4i+1} = \frac{4i-5}{-4i-3} = \frac{5-4i}{4i+3}\cdot\frac{4i-3}{4i-3} = \frac{20i-15+16+12i}{-25} = \frac{32i+1}{-25}$$

12. $(1-i\sqrt{3})^6(1+i)^{10} = 2^6\cdot 2^5 i = 2^{11} i$

$z_1 = 1-i\sqrt{3} = 2\cdot e^{-\frac{\pi}{3}i}$ $\qquad z_1^6 = 2^6\cdot e^{-2\pi i} = 2^6\cdot e^{2\pi i}$

$|z_1| = \sqrt{1+3} = 2$ $\qquad = 2^6\cdot(1+0i)$

$tg\varphi = \frac{y}{x} = -\sqrt{3}$ $\qquad = 2^6$

$\arg z_1 = -\frac{\pi}{3}$

$z_2 = 1+i = \sqrt{2}\, e^{\frac{\pi}{4}i}$ $\qquad z_2^{10} = 2^5\cdot e^{\frac{10\pi}{4}i} = 2^5\cdot e^{(2\pi+\frac{\pi}{2})i}$

$|z_2| = \sqrt{2}$ $\qquad = 2^5\cdot(0+i)$

$tg\varphi = \frac{y}{x} = 1$ $\qquad = 2^5 i$

$\arg z_2 = \frac{\pi}{4}$

13. a) $\sqrt[6]{1}$

$z = 1 + 0i = (\cos(0+2k\pi) + i\sin(0+2k\pi))$

$|z| = \sqrt{1} = 1$

$tg\varphi = \frac{y}{x} = 0$

$\arg z = 0 + 2k\pi$

$\sqrt[6]{z} = \sqrt[6]{1}\left(\cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6}\right)$

$= \left(\cos\frac{k\pi}{3} + i\sin\frac{k\pi}{3}\right) \quad k = 0, 1, 2, 3, 4, 5$

$z_0 = \cos 0 + i\sin 0 = 1$

$z_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$z_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$z_3 = \cos\pi + i\sin\pi = -1$

$z_4 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$z_5 = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

b) $\sqrt[3]{-8i}$

$z = 0 - 8i = 8\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

$|z| = \sqrt{64} = 8$

$tg\varphi = \frac{y}{x} = \frac{-8}{0} = \infty$

$\arg z = -\frac{\pi}{2}$

$\sqrt[3]{z} = \sqrt[3]{8}\left(\cos\frac{-\frac{\pi}{2}+2k\pi}{3} + i\sin\frac{-\frac{\pi}{2}+2k\pi}{3}\right)$

$= 2\left(\cos\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right) + i\sin\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right)$

$k = 0, 1, 2$

$z_0 = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \sqrt{3} - i$

$z_1 = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2(0 + i) = 2i$

$z_2 = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = -\sqrt{3} - i$

c) $\sqrt[3]{-1+i}$

$\arg\varphi = \frac{3\pi}{4}$

$z = -1 + i$

$|z| = \sqrt{2}$

$tg\varphi = \frac{y}{x} = -1$

$\sqrt[3]{z} = \sqrt[6]{2}\left(\cos\frac{\frac{3\pi}{4}+2k\pi}{3} + i\sin\frac{\frac{3\pi}{4}+2k\pi}{3}\right)$

$= \sqrt[6]{2}\left(\cos\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{2k\pi}{3}\right)\right)$

$$z_0 = \sqrt[6]{2}\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) = \sqrt[6]{2}\cdot\frac{\sqrt{2}}{2}\,(1+i)$$
$$= \frac{1}{\sqrt[3]{2}}\,(1+i)$$

$$z_1 = \sqrt[6]{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$$

$$z_2 = \sqrt[6]{2}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$$