10. $1 : 150 = x : 30\,m$

$$x = \frac{30}{150}$$

$$x = 0{,}2\,m$$

11. $1 : x = 8 : 4000$

$$x = \frac{4000}{8}$$

$$x = 500$$

Razmera je $1:500$.

2. Realne funkcije - granične vrednosti i izvodi

2.1. Osnovni pojmovi o preslikavanju

Ako svakom elementu skupa A (originalima) dodelimo (pridružimo) tačno jedan element skupa B (sliku) kažemo da smo izvršili preslikavanje skupa A u skup B. Skup A je domen, skup B kodomen preslikavanja (funkcije).

Neka funkcija f preslikava skup A u skup B ($f: A \to B$). Ovo preslikavanje je injektivno ("1-1"), ako svakoj slici odgovara jedan original. Posmatrano preslikavanje je sirjektivno ("na") ako se svi elementi skupa B pojavljuju kao slike.

a) $x^2 + y^2 = 2$ D: $2 - x^2 \geq 0$ K: $y_1 \in [0, \sqrt{2}]$

$y^2 = 2 - x^2$ $x^2 \leq 2$ $y_2 \in [-\sqrt{2}, 0]$

$y_1 = \sqrt{2 - x^2}$ $x \in D = [-\sqrt{2}, \sqrt{2}]$

$y_2 = -\sqrt{2 - x^2}$

$y_1 = \sqrt{2-x^2} : [-\sqrt{2}, \sqrt{2}] \rightarrow [0, \sqrt{2}]$

$y_2 = -\sqrt{2-x^2} : [-\sqrt{2}, \sqrt{2}] \rightarrow [-\sqrt{2}, 0]$

b) $y^2 = -2x$ $\qquad y_1 = \sqrt{-2x} \qquad K: y_1 \in [0, \infty)$

$y = \pm\sqrt{-2x} \qquad D: -2x \geq 0$

$x \leq 0 \qquad y_2 \in (-\infty, 0]$

$y_2 = -\sqrt{-2x}$

$D: -2x \geq 0$

$x \leq 0$

$\sqrt{-2x} : (-\infty, 0] \rightarrow [0, \infty) \qquad -\sqrt{-2x} : (-\infty, 0] \rightarrow (-\infty, 0]$

c) $x = \sin y - 1 \qquad D = [-2, 0]$

$\sin y = x+1 \qquad K = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$y = \arcsin(x+1)$

12.2. Nizovi

- Niz je funkcija a $N \rightarrow R$, a_n je oznaka za opšti član niza

- Ako u svakoj ε-okolini $(A-\varepsilon, A+\varepsilon)$ tačke A postoji beskonačno mnogo članova niza a_n, kažemo da je tačka A tačka nagomilavanja niza.

- Ako u svakoj ε-okolini $(A-\varepsilon, A+\varepsilon)$ tačke A postoji beskonačno mnogo članova niza a_n, a van nje konačno mnogo, odnosno ako

$\forall \varepsilon > 0, \exists n_0 \in N, \forall n \geq n_0, |A - a_n| < \varepsilon$

...žemo da je A granična vrednost niza a_n i pišemo $A = \lim_{n\to\infty} a_n$

Niz je ograničen sa gornje (donje) strane ako postoji M (m) tako da za ...e $n \in N$ važi $a_n \leq M$ ($a_n \geq m$)

Niz je monotono rastući (opadajući, nerastući, neopadajući) ako za sve ...$\in N$ važi $a_{n+1} > a_n$ ($a_{n+1} < a_n$, $a_{n+1} \leq a_n$, $a_{n+1} \geq a_n$).

Osobine granične vrednosti niza

...a konvergentne nizove a_n i b_n i brojeve $\alpha, \beta \in R$ važi

$\lim_{n\to\infty} (\alpha a_n + \beta b_n) = \alpha \lim_{n\to\infty} a_n + \beta \lim_{n\to\infty} b_n$

$\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, $\lim_{n\to\infty} b_n \neq 0$

...ko je funkcija f neprekidna u tački $\lim_{n\to\infty} a_n$ onda je

$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$

...ažne granične vrednosti

$$\lim_{n\to\infty} \frac{1}{n^\alpha} = \begin{cases} 0, & \alpha > 0 \\ 1, & \alpha = 0 \\ +\infty, & \alpha < 0 \end{cases} \qquad \lim_{n\to\infty} q^n = \begin{cases} 0, & |q| < 1 \\ 1, & q = 1 \\ +\infty, & q > 1 \end{cases}$$

...određeni oblici

"$\infty \cdot \infty = \infty$", "$\infty + \infty = \infty$", "$\infty^\infty = \infty$", "$\frac{1}{\pm 0} = \pm\infty$"

"$\frac{\cdot}{\infty} = 0$", "$\frac{0}{\pm\infty} = 0$", "$0^\infty = 0$", "$\frac{\infty}{\pm 0} = \pm\infty$",

"arctg $\pm\infty = \pm\frac{\pi}{2}$", "$\ln 0 = -\infty$", "$\ln \infty = \infty$",

"$2^{\infty}=\infty$", "$3^{-\infty}=0$" "$\ln 1=0$"

- Neodređeni oblici

"$\infty-\infty$", "$0\cdot\infty$", "$\frac{\infty}{\infty}$", "1^{∞}", "0^{0}", "∞^{0}".

Znaci navoda u gornjim izrazima stoje u smislu granične vrednosti.

Npr. "$0\cdot\infty$" $\overset{def.}{=} \lim\limits_{a\to 0,\, b\to\infty} a\cdot b$

2. a) $a_n=n^2+1$

$a_1=2,\ a_2=5,\ a_3=10,\ a_4=17\ldots$

- Niz je monotono rastući: $((n+1)^2+1>n^2+1)$
- Ograničen samo s donje strane
- Bez tačke nagomilavanja
- Nema graničnu vrednost $\lim\limits_{n\to\infty}(n^2+1)=+\infty$. Posmatrani niz divergira ka $+\infty$.

b) $b_n=\frac{n-1}{n}$

$b_1=0,\ b_2=\frac{1}{2},\ b_3=\frac{2}{3},\ b_4=\frac{3}{4}\ldots$

- Niz je monotono rastući: $b_n<b_{n+1}$

$$\frac{n-1}{n}<\frac{n}{n+1}\quad \Big|\,(n+1)$$

$$n^2-1<n^2 \quad T$$

- Niz je ograničen $0\le n-1<n$

$$0\le\frac{n-1}{n}<1$$

- Niz je konvergentan $\lim\limits_{n\to\infty}\frac{n-1}{n}=\frac{\cancel{n}\cdot\left(1-\overset{\to 0}{\frac{1}{n}}\right)}{\cancel{n}}=1$

Za proizvoljno $\varepsilon > 0 \Rightarrow \left|\frac{n-1}{n} - 1\right| = \left|\frac{\cancel{n} - 1 - \cancel{n}}{n}\right| = \frac{1}{n} < \varepsilon \quad \forall n > \frac{1}{\varepsilon}$

Broj 1 je tačka nagomilavanja

c) $c_n = \frac{(-1)^{n-1}}{n}$

$c_1 = 1,\ c_2 = -\frac{1}{2},\ c_3 = \frac{1}{3},\ c_4 = -\frac{1}{4}, \dots$

Niz nije monoton, ali je ograničen

$|c_n| = \frac{1}{n} \leq 1,\ \forall n \in N$

Niz je konvergentan $\lim\limits_{n\to\infty} \frac{(-1)^{n-1}}{n} = 0$

Broj 0 je tačka nagomilavanja.

d) $d_n = (-1)^{n-1} \cdot \frac{2n-1}{2n}$

$d_1 = \frac{1}{2},\ d_2 = -\frac{3}{4},\ d_3 = \frac{5}{6},\ d_4 = -\frac{7}{8}$

Niz je oscilatoran

$|d_n| = \frac{2n-1}{2n} < 1$, niz je ograničen

Ima dve tačke nagomilavanja: -1 i 1.

Parni članovi se gomilaju oko -1, a neparni oko 1.

e) $e_n = n^{(-1)^{n-1}}$

$e_1 = 1,\ e_2 = \frac{1}{2},\ e_3 = 3,\ e_4 = \frac{1}{4},\ e_5 = 5 \dots$

Niz nije monoton.

Nije ograničen, pa ne može biti konvergentan.

Tačka nagomilavanja je 0, oko nje se grupišu parni članovi niza. To je jedina tačka nagomilavanja.

f) $f_n = \ln n$

$f_1 = 0,\ f_2 = \ln 2,\ f_3 = \ln 3,\ f_4 = \ln 4$

- Niz je monotono rastući
- Neograničen je, bez tačaka nagomilavanja.
- Niz je divergentan i $\lim_{n\to\infty} \ln n = +\infty$

g) $g_n = \left(\frac{1}{2}\right)^n$

$g_1 = \frac{1}{2},\ g_2 = \frac{1}{4},\ g_3 = \frac{1}{8},\ g_4 = \frac{1}{16} \dots$

- Niz je monotono opadajući jer je $\left(\frac{1}{2}\right)^n > \left(\frac{1}{2}\right)^{n+1} \iff 1 > \frac{1}{2} \iff T.$
- Zbog $0 < \left(\frac{1}{2}\right)^n \leq 1$ on je i ograničen.
- Niz je konvergentan $\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$
- Broj 0 je jedina tačka nagomilavanja.

h) $h_n = \sin\frac{n\pi}{2}$

$h_1 = 1,\ h_2 = 0,\ h_3 = -1,\ h_4 = 0,\ h_5 = 1,\ h_6 = 0,\ h_7 = -1 \dots$

- Niz nije monoton.
- Ograničen je. Kao elementi ovog niza pojavljuju se samo 3 broja: $-1, 0, 1$.
- $-1, 0, 1$ su tačke nagomilavanja.
- Niz koji ima više tačaka nagomilavanja ne može imati graničnu vrednost

i) $i_n = \begin{cases} n, & n \leq 4 \\ 5, & n > 4 \end{cases}$

$i_1 = 1,\ i_2 = 2,\ i_3 = 3,\ i_4 = 4,\ i_5 = 5,\ i_6 = 5,\ i_7 = 5 \dots$

- Niz je monotono neopadajući. Ako zanemarimo prva 4 člana, niz je stacionaran čija je granična vrednost i jedina tačka nagomilavanja:

$\lim_{n\to\infty} i_n = 5$

3. a) $\lim\limits_{n\to\infty} \frac{100n}{n^2+1} = \lim\limits_{n\to\infty} \frac{\cancel{n}\cdot 100}{n^{\cancel{2}}\left(1+\frac{1}{n^2}\right)} = \lim\limits_{n\to\infty} \frac{100}{n} = 0$

b) $\lim\limits_{n\to\infty} \frac{3n^3+6n-1}{2n^3+3n^2} = \lim\limits_{n\to\infty} \frac{\cancel{n^3}\left(3+\frac{6}{n^2}-\frac{1}{n^3}\right)}{\cancel{n^3}\left(2+\frac{3}{n}\right)} = \frac{3}{2}$

c) $\lim\limits_{n\to\infty} \frac{n^2}{8n+100} = \lim\limits_{n\to\infty} \frac{n^{\cancel{2}}}{\cancel{n}\left(8+\frac{100}{n}\right)} = \lim\limits_{n\to\infty} \frac{n}{8} = \infty$

d) $\lim\limits_{n\to\infty} \frac{\sqrt[3]{n^2}+2n+3}{n^2+2n+1} = \lim\limits_{n\to\infty} \frac{\cancel{n^2}\cdot\left(\frac{1}{n^{\frac{4}{3}}}+\frac{2}{n}+\frac{3}{n^2}\right)}{\cancel{n^2}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)} = \frac{0}{1} = 0$

e) $\lim\limits_{n\to\infty} \frac{2^n+(-3)^{n+1}}{\left(\frac{3}{2}\right)^{n+2}-4^n} = \lim\limits_{n\to\infty} \frac{\cancel{4^n}\cdot\left(\left(\frac{1}{2}\right)^n+(-3)\cdot\left(-\frac{3}{4}\right)^n\right)}{\cancel{4^n}\cdot\left(\left(\frac{3}{2}\right)^2\cdot\left(\frac{3}{8}\right)^n-1\right)} = \frac{0}{-1} = 0$

f) $\lim\limits_{n\to\infty} \frac{5^{n+2}-3^n}{(-4)^{n+1}-5^n} = \lim\limits_{n\to\infty} \frac{\cancel{5^n}\left(5^2\cdot 1-\left(\frac{3}{5}\right)^n\right)}{\cancel{5^n}\left(-4\cdot\left(-\frac{4}{5}\right)^n-1\right)} = \frac{25}{-1} = -25$

g) $\lim\limits_{n\to\infty} \frac{5^n+(-2)^n}{3^{n+2}+5} = \lim\limits_{n\to\infty} \frac{\cancel{5^n}\cdot\left(1+\left(-\frac{2}{5}\right)^n\right)}{\cancel{5^n}\left(3^2\cdot\left(\frac{3}{5}\right)^n+\left(\frac{5}{5^n}\right)\right)} = \frac{1}{0} = \infty$

h) $\lim\limits_{n\to\infty} \frac{2^n+n^5}{5^n-\left(\frac{3}{2}\right)^{n+1}} = \lim\limits_{n\to\infty} \frac{\cancel{5^n}\left(\left(\frac{2}{5}\right)^n+\frac{n^5}{5^n}\right)}{\cancel{5^n}\left(1-\left(\frac{3}{2}\right)\cdot\left(\frac{3}{10}\right)^n\right)} = \frac{0}{1} = 0$

4. $\lim\limits_{n\to\infty} \left(\sqrt{n^2+4n+1}-\sqrt{n^2+n}\right)\cdot\frac{\sqrt{n^2+4n+1}+\sqrt{n^2+n}}{\sqrt{n^2+4n+1}+\sqrt{n^2+n}} =$

$= \lim\limits_{n\to\infty} \frac{\cancel{n^2}+4n+1-\cancel{n^2}-n}{\sqrt{n^2+4n+1}+\sqrt{n^2+n}} = \lim\limits_{n\to\infty} \frac{3n+1}{\sqrt{n^2+4n+1}+\sqrt{n^2+n}}$

$\lim\limits_{n\to\infty} \frac{\cancel{n}\cdot\left(3+\frac{1}{n}\right)}{\cancel{n}\left(\sqrt{1+\frac{4}{n}+\frac{1}{n^2}}+\sqrt{1+\frac{1}{n}}\right)} = \frac{3}{2}$

b) $\lim\limits_{n\to\infty} \left(\sqrt[3]{n^3+n^2}-\sqrt[3]{n^3-1}\right)\cdot\frac{\left(\sqrt[3]{n^3+n^2}\right)^2+\left(\sqrt[3]{n^3+n^2}\cdot\sqrt[3]{n^3-1}\right)+\left(\sqrt[3]{n^3-1}\right)^2}{\left(\sqrt[3]{n^3+n^2}\right)^2+\left(\sqrt[3]{n^3+n^2}\cdot\sqrt[3]{n^3-1}\right)+\left(\sqrt[3]{n^3-1}\right)^2} =$

$\lim\limits_{n\to\infty} \frac{\cancel{n^3}+n^2-\cancel{n^3}+1}{\left(\sqrt[3]{n^3+n^2}\right)^2+\sqrt[3]{(n^3+n^2)(n^3-1)}+\left(\sqrt[3]{n^3-1}\right)^2} =$

$$\lim_{n\to\infty} \frac{n^2+1}{\sqrt[3]{n^6+2n^5+n^4}+\sqrt[3]{n^6+n^5-n^3-n^2}+\sqrt[3]{n^6-2n^3+1}} =$$

$$= \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n^2}\right)}{n^2\left(\sqrt[3]{1+\frac{2}{n}+\frac{1}{n^2}}+\sqrt[3]{1+\frac{1}{n}-\frac{1}{n^3}-\frac{1}{n^4}}+\sqrt[3]{1-\frac{2}{n^3}+\frac{1}{n^6}}\right)} =$$

$$= \frac{1}{3}$$

12.3 Granične vrednosti funkcije

- Definicija granične vrednosti funkcije u tački a: $\lim_{x\to a} f(x) = l$
$\forall \varepsilon > 0\ \exists \delta > 0\ (0 < |x-a| < \delta \Rightarrow |f(x) - l| < \varepsilon)$
- Pri čemu funkcija $f(x)$ mora biti definisana u nekoj okolini tačke a, a u samoj tački a ne mora

- Na sličan način se definišu i granične vrednosti kad $x \to \pm\infty$ kao i jednostrane granične vrednosti

- Osnovne osobine graničnih vrednosti funkcije su iste kao osobine graničnih vrednosti nizova, a važe isti određeni i neodređeni oblici.

- Osnovne osobine graničnih vrednosti

$$\lim_{x\to a} (\alpha f(x) + \beta g(x)) = \alpha \lim_{x\to a} f(x) + \beta \lim_{x\to a} g(x)$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \quad \lim_{x\to a} g(x) \neq 0$$

$$\lim_{x\to a} (f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

Ako je funkcija neprekidna u tački $\lim_{x\to a} f(x)$ onda je
$\lim_{x\to a} h(f(x)) = h(\lim_{x\to a} f(x))$,
pod pretpostavkom da navedene granične vrednosti postoje.

Važne granične vrednosti funkcija

$$\lim_{x\to +\infty} \frac{1}{x^\alpha} = \begin{cases} 0, & \alpha > 0 \\ 1, & \alpha = 0 \\ +\infty, & \alpha < 0 \end{cases} \qquad \lim_{x\to +\infty} q^x = \begin{cases} 0, & q \in (0,1) \\ 1, & q = 1 \\ +\infty, & q > 1 \end{cases}$$

5.

a) $\lim_{x\to\infty} \frac{\sqrt[3]{2x^2} - \sqrt[3]{x} + 1}{\sqrt[3]{x^2 + 2x + 1}} = \lim_{x\to\infty} \frac{\cancel{\sqrt[3]{x^2}} \cdot \left(\sqrt[3]{2} - \frac{1}{\sqrt[3]{x}} + \frac{1}{\sqrt[3]{x^2}}\right)}{\cancel{\sqrt[3]{x^2}} \cdot \left(\sqrt[3]{1 + \frac{2}{x} + \frac{1}{x^2}}\right)} = \sqrt[3]{2}$

b) $\lim_{x\to -\infty} \frac{x^4 - x^3}{\sqrt[3]{x^2 + 2^x}} = \lim_{x\to -\infty} \frac{\cancel{x^4} \cdot \left(1 - \frac{1}{x}\right)}{\cancel{x^4} \cdot \left(\sqrt[3]{\frac{1}{x^{10}} + \frac{2^x}{x^4}}\right)} = \frac{1}{0} = \infty$

a) $\lim_{x\to 0} \frac{x^2 + x^3 - 2x^4}{2x^2 + 3x^5} = \lim_{x\to 0} \frac{\cancel{x^2}(1 + x - 2x)}{\cancel{x^2}(2 + 3x^3)} = \frac{1+0-0}{2+0} = \frac{1}{2}$

b) $\lim_{x\to 0} \frac{(1+x)(1+3x) - 1}{2x} = \lim_{x\to 0} \frac{\cancel{1} + 3x + x + 3x^2 - \cancel{1}}{2x} = \lim_{x\to 0} \frac{3x^2 + 4x}{2x} =$

$= \lim_{x\to 0} \frac{\cancel{x}(3x+4)}{2\cancel{x}} = \frac{0+4}{2} = 2$

a) $\lim_{x\to 1} \frac{x^2 - 1}{2x^2 - x - 1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{2\cancel{(x-1)}(x + \frac{1}{2})} =$ $\qquad x_{1,2} = \frac{1 \pm \sqrt{1+8}^{\,3}}{4} \nearrow x_1 = 1, \searrow x_2 = -\frac{1}{2}$

$= \lim_{x\to 1} \frac{x+1}{2(x + \frac{1}{2})} = \frac{2}{3}$

) $\lim_{x\to -3} \frac{x^2 - x - 12}{2x^2 + 11x + 15} =$ $\qquad x_{1,2} = \frac{1 \pm \sqrt{1+48}^{\,7}}{2} \nearrow x_1 = 4, \searrow x_2 = -3$

$\lim_{x\to -3} \frac{(x-4)\cancel{(x+3)}}{2\cancel{(x+3)}(x + \frac{5}{2})} =$ $\qquad x_{1,2} = \frac{-11 \pm \sqrt{121 - 120}^{\,1}}{4} \nearrow x_1 = -3, \searrow x_2 = -\frac{5}{2}$

$$= \lim_{x \to -3} \frac{x-4}{2x+5} = \frac{-7}{6+5} = 7$$

c) $$\lim_{x \to 2} \frac{x^3+2x^2-4x-8}{x^3-x^2-x-2} =$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x^2+4x+4)}{\cancel{(x-2)}(x^2+x+1)} =$$

$$= \lim_{x \to 2} \frac{x^2+4x+4}{x^2+x+1} =$$

$$= \frac{4+8+4}{4+2+1} =$$

$$= \frac{16}{7}$$

$$(\cancel{x^3}+2x^2-4x-8):(x-2) = x^2+4x+4$$
$$\underline{\cancel{-x^3}+2x^2}$$
$$\cancel{4x^2}-4x$$
$$\underline{\cancel{-4x^2}+8x}$$
$$4x-8$$
$$\underline{-4x+8}$$
$$//$$

$$(\cancel{x^3}-x^2-x-2):(x-2) = x^2+x+1$$
$$\underline{\cancel{-x^3}+2x^2}$$
$$\cancel{x^2}-x$$
$$\underline{\cancel{-x^2}+2x}$$
$$x-2$$
$$\underline{-x+2}$$
$$//$$

d) $$\lim_{x \to 1} \frac{x^3-x^2-x+1}{x^4-x^3-x+1} = \lim_{x \to 1} \frac{x^2(x-1)-(x-1)}{x^3(x-1)-(x-1)} = \lim_{x \to 1} \frac{\cancel{(x-1)}(x^2-1)}{\cancel{(x-1)}(x^3-1)} = \lim_{x \to 1} \frac{x^2-1}{x^3-1} =$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x^2+x+1)} = \lim_{x \to 1} \frac{x+1}{x^2+x+1} = \frac{2}{3}$$

e) $$\lim_{x \to 2} \frac{x^3+2x^2-5x-6}{x^3-3x^2+4} =$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x^2+4x+3)}{\cancel{(x-2)}(x^2-x-2)} =$$

$$= \lim_{x \to 2} \frac{x^2+4x+3}{x^2-x-2} =$$

$$= \frac{4+8+3}{4-2-2} = \frac{15}{0} = \infty$$

$$(\cancel{x^3}+2x^2-5x-6):(x-2) = x^2+4x+3$$
$$\underline{\cancel{-x^3}+2x^2}$$
$$\cancel{4x^2}-5x$$
$$\underline{\cancel{-4x^2}+8x}$$
$$3x-6$$
$$\underline{-3x+6}$$
$$//$$

$$(\cancel{x^3}-3x^2+4):(x-2) = x^2-x-2$$
$$\underline{\cancel{-x^3}+2x^2}$$
$$\cancel{-x^2}+4$$
$$\underline{\cancel{x^2}-2x}$$
$$-2x+4$$
$$\underline{2x-4}$$
$$//$$

- Ova granična vrednost ne postoji jer brojilac teži ka 15, a imenilac ka 0.

Treba posmatrati jednostrane granične vrednosti:

$$\lim_{x\to 2+0}\frac{x^2+4x+3}{x^2-x-2}= \qquad x_{1,2}=\frac{-4\pm\sqrt{16-12}^{\,2}}{2}\to x_1=-3,\; x_2=-1$$

$$=\lim_{x\to 2+0}\frac{(x+3)(x+1)}{(x-2)(x+1)}= \qquad x_{1,2}=\frac{1\pm\sqrt{1+8}^{\,3}}{2}\to x_1=-1,\; x_2=2$$

$$=\frac{3\cdot 5}{3\cdot(+0)}=+\infty$$

$$\lim_{x\to 2-0}\frac{x^2+4x+3}{x^2-x-2}=\lim_{x\to 2-0}\frac{(x+1)(x+3)}{(x+1)(x-2)}=\frac{3\cdot 5}{3(-0)}=-\infty$$

2. a) $\lim\limits_{x\to 3}\dfrac{\sqrt{x+6}-x}{x-3}\cdot\dfrac{\sqrt{x+6}+x}{\sqrt{x+6}+x}=$

$$=\lim_{x\to 3}\frac{x+6-x^2}{(x-3)(\sqrt{x+6}+x)}= \qquad x_{1,2}=\frac{-1\pm\sqrt{1+24}^{\,5}}{-2}\to x_1=3,\; x_2=-2$$

$$=\lim_{x\to 3}\frac{-(x+2)\cancel{(x-3)}}{\cancel{(x-3)}(\sqrt{x+6}+x)}=$$

$$=\lim_{x\to 3}\frac{-x-2}{\sqrt{x+6}+x}=\frac{-5}{6}$$

b) $\lim\limits_{x\to 4}\dfrac{\sqrt{1+2x}-3}{\sqrt{x}-2}\cdot\dfrac{\sqrt{1+2x}+3}{\sqrt{1+2x}+3}=$

$$=\lim_{x\to 4}\frac{1+2x-9}{(\sqrt{x}-2)(\sqrt{1+2x}+3)}=$$

$$\lim_{x\to 4}\frac{2(x-4)}{(\sqrt{x}-2)(\sqrt{1+2x}+3)}=$$

$$\lim_{x\to 4}\frac{2\cancel{(\sqrt{x}-2)}(\sqrt{x}+2)}{\cancel{(\sqrt{x}-2)}(\sqrt{1+2x}+3)}=$$

$$\lim_{x\to 4}\frac{2(\sqrt{x}+2)}{(\sqrt{1+2x}+3)}=\frac{2\cdot 4}{3+3}=\frac{8}{6}=\frac{4}{3}$$

12. 4. Izvod funkcije

- Neka je funkcija $f(x)$ definisana u nekoj okolini tačke x. Ako postoji granična vrednost $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$, tada ovu graničnu vrednost nazivamo prvi izvod funkcije $f(x)$ u tački x i obeležavamo sa $f'(x)$. Za funkciju f tada kažemo da je diferencijabilna u tački x.

- Viši izvodi $f^{(n+1)}(x) = \left(f^{(n)}(x)\right)'$, $n \in N$

- Za diferencijabilne funkcije f i g i brojeve α i β važi:

1. $(\alpha f(x) + \beta g(x))' = \alpha f'(x) + \beta g'(x)$
2. $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
3. $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}$, $g(x) \neq 0$
4. $(f(g(x)))' = f'(g(x))\, g'(x)$

* Tablica prvih izvoda

I) $c = 0$

II) $(x^n)' = n x^{n-1}$, $n \neq 0$

III) $(\log_a x)' = \frac{1}{x \ln a}$

IV) $(a^x)' = a^x \ln a$

V) $(\sin x)' = \cos x$

VI) $(\cos x)' = -\sin x$

VII) $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$

VIII) $(\operatorname{arctg} x)' = \frac{1}{1+x^2}$

IX) $(\ln x)' = \frac{1}{x}$

X) $(e^x)' = e^x$

9. a) $(cx)' = c$

$$\lim_{\Delta x \to 0} \frac{c(x+\Delta x) - cx}{\Delta x} = \lim_{\Delta x \to 0} \frac{c\Delta x}{\Delta x} = c$$

b) $(x^2 - x + 1)' = 2x - 1$

$$\lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - (x+\Delta x) + 1 - (x^2 - x + 1)}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x\Delta x + \Delta x^2 - \cancel{x} - \Delta x + \cancel{1} - \cancel{x^2} + \cancel{x} - \cancel{1}}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \left(\frac{\Delta x (2x-1)}{\Delta x} + \Delta x \right) = 2x - 1$$

10. a) $y = x^2 \cos x + \sin x$

$$y' = (x^2 \cos x)' + (\sin x)'$$

$$= (x^2)' \cos x + x^2 (\cos x)' + \cos x$$

$$= 2x \cos x - x^2 \sin x + \cos x$$

$$= (2x+1)\cos x - x^2 \sin x$$

b) $y = \frac{e^x}{x^2} + 2x \ln x$

$$y' = \left(\frac{e^x}{x^2}\right)' + (2x \ln x)'$$

$$= \frac{(e^x)' \cdot x^2 - e^x (x^2)'}{x^4} + (2x)' \cdot \ln x + 2x (\ln x)'$$

$$= \frac{e^x \cdot x^2 - 2x e^x}{x^4} + 2\ln x + 2\cancel{x} \cdot \frac{1}{\cancel{x}}$$

$$= \frac{\cancel{x} e^x (x-2)}{x^{\cancel{4}\,3}} + 2\ln x + 2 =$$

$$= \frac{x-2}{x^3} e^x + 2\ln x + 2$$

c) $y = \operatorname{tg} x$

$$y' = \left(\frac{\sin x}{\cos x}\right)'$$

$$= \frac{(\sin x)' \cdot \cos x - \sin x (\cos x)'}{\cos^2 x}$$

$$= \frac{\cos x \cdot \cos x + \sin x \cdot \sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

d) $y = \ln tg\left(\frac{x}{2} + \frac{\pi}{4}\right)$

$$y' = \frac{1}{tg\left(\frac{x}{2} - \frac{\pi}{4}\right)} \cdot \left(tg\left(\frac{x}{2} + \frac{\pi}{4}\right)\right)'$$

$$= \frac{\cos\left(\frac{x}{2} - \frac{\pi}{4}\right)}{\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)} \cdot \left(\frac{\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)}{\cos\left(\frac{x}{2} + \frac{\pi}{4}\right)}\right)'$$

$$= \frac{\cancel{\cos\left(\frac{x}{2} + \frac{\pi}{4}\right)}}{\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)} \cdot \frac{\left(\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)\right)' \cdot \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) - \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) \cdot \left(\cos\left(\frac{x}{2} + \frac{\pi}{4}\right)\right)'}{\cos\left(\frac{x}{2} + \frac{\pi}{4}\right)^{\cancel{2}}} =$$

$$= \frac{\cos\left(\frac{x}{2} + \frac{\pi}{4}\right) \cdot \left(\frac{x}{2} + \frac{\pi}{4}\right)' \cdot \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) + \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) \cdot \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) \cdot \left(\frac{x}{2} + \frac{\pi}{4}\right)'}{\sin\left(\frac{x}{2} + \frac{\pi}{4}\right) \cos\left(\frac{x}{2} + \frac{\pi}{4}\right)}$$

$$= \frac{\cos^2\left(\frac{x}{2} + \frac{\pi}{4}\right) \cdot \frac{1}{2} + \sin^2\left(\frac{x}{2} + \frac{\pi}{4}\right) \cdot \frac{1}{2}}{\sin\left(\frac{x}{2} + \frac{\pi}{4}\right) \cdot \cos\left(\frac{x}{2} + \frac{\pi}{4}\right)} =$$

$$= \frac{1}{\sin\left(\frac{x}{2} - \frac{\pi}{4}\right) \cdot \cos\left(\frac{x}{2} + \frac{\pi}{4}\right)} \cdot \frac{1}{2} = \frac{1}{2 \sin\left(\frac{x}{2} + \frac{\pi}{4}\right) \cos\left(\frac{x}{2} + \frac{\pi}{4}\right)} =$$

$$= \frac{1}{\sin\left(x + \frac{\pi}{2}\right)} = \frac{1}{\cos x}$$

11. $y = f(x)$

a) $y = \ln\left(x + \sqrt{1 + x^2}\right)$

$$y' = \frac{1}{x + \sqrt{1 + x^2}} \cdot \left(x + \sqrt{1 + x^2}\right)'$$

$$= \frac{1}{x + \sqrt{1 + x^2}} \cdot \left(1 + \frac{1}{2\sqrt{1 + x^2}} \cdot \left(1 + x^2\right)'\right) =$$

$$= \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1+\frac{1}{\cancel{2}\sqrt{1+x^2}} \cdot \cancel{2}x\right) =$$

$$= \frac{1}{\cancel{x+\sqrt{x^2+1}}} \cdot \frac{\cancel{\sqrt{1+x^2}+x}}{\sqrt{1+x^2}} =$$

$$= \frac{1}{\sqrt{1+x^2}}$$

b) $y = \ln \frac{x}{1+\sqrt{1+x^2}}$

$$y' = \frac{1+\sqrt{1+x^2}}{x} \cdot \left(\frac{x}{1+\sqrt{1+x^2}}\right)' =$$

$$= \frac{\cancel{1+\sqrt{1+x^2}}}{x} \cdot \frac{(x)'(1+\sqrt{1+x^2}) - x\left(1+\sqrt{1+x^2}\right)'}{\left(1+\sqrt{1+x^2}\right)^{\cancel{2}}} =$$

$$= \frac{(1+\sqrt{1+x^2}) - x \cdot \frac{1}{2\sqrt{1+x^2}} \cdot (1+x^2)'}{x\left(1+\sqrt{1+x^2}\right)}$$

$$= \frac{1+\sqrt{1+x^2} - \frac{\cancel{2}x^2}{\cancel{2}\sqrt{1+x^2}}}{x\left(1+\sqrt{1+x^2}\right)} =$$

$$= \frac{\frac{(1+\sqrt{1+x^2}) \cdot \sqrt{1+x^2} - x^2}{\sqrt{1+x^2}}}{x\left(1+\sqrt{1+x^2}\right)} = \frac{\frac{\sqrt{1+x^2} + 1 + \cancel{x^2} - \cancel{x^2}}{\sqrt{1+x^2}}}{x\left(1+\sqrt{1+x^2}\right)} = \frac{\cancel{\sqrt{1+x^2}+1}}{\sqrt{1+x^2} \cdot x \cdot \cancel{\left(1+\sqrt{1+x^2}\right)}} =$$

$$= \frac{1}{x\sqrt{1+x^2}}$$

) $y = \frac{x}{1+x^2} + \operatorname{arctg} x$

$$y' = \left(\frac{x}{1+x^2}\right)' + (\operatorname{arctg} x)' =$$

$$= \frac{(x)' \cdot (1+x^2) - x(1+x^2)'}{(1+x^2)^2} + \frac{1}{1+x^2} =$$

$$= \frac{1+x^2-2x^2}{(1+x^2)^2} + \frac{1}{1+x^2} = \frac{1-x^2}{(1+x^2)^2} + \frac{1}{1+x^2} = \frac{1-\cancel{x^2}+1+\cancel{x^2}}{(1+x^2)^2} = \frac{2}{(1+x^2)^2}$$

d) $y = x^x$

$y' = (x^x)'$

$= (e^{\ln x^x})'$

$= (e^{x\ln x})'$

$= e^{x\ln x} \cdot (x\ln x)'$

$= e^{x\ln x} \cdot ((x)' \cdot \ln x + x \cdot (\ln x)')$

$= e^{x\ln x} \cdot (\ln x + \cancel{x} \cdot \frac{1}{\cancel{x}})$

$= e^{x\ln x} \cdot (\ln x + 1)$

$= x^x (1 + \ln x)$

- Prvi izvod funkcije u nekoj tački predstavlja koeficijent pravca tangente u posmatranoj tački.

- Jednačina tangente na krivu $f(x)$ u tački $M(x_0, y_0)$, $y_0 = f(x_0)$, glasi $y - y_0 = f'(x_0)(x - x_0)$, a jednačina normale $y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

- Ako presek x-ose sa tangentom označimo sa T, a presek sa normalom sa N, dužina tangente je jednaka $t = |\vec{MT}|$, a dužina normale $n = |\vec{MN}|$. Projekcija tačke M na x-osu je $P(x_0, 0)$. Duži TP i NP nazivamo subtangenta i subnormala. Dužine subtangente i subnormale možemo izračunati kao $s_t = |\vec{TP}|$ i $s_n = |\vec{NP}|$.

12. a) $y = x^3 + x + 1$

$M(1, y_0)$

$y' = (x^3 + x + 1)' = 3x^2 + 1$

$x_0 = 1 \quad y_0 = y(x_0) = 1^3 + 1 + 1 = 3$

$y_0'(x_0) = 3 \cdot 1^2 + 1 = 4$

$t: y - 3 = 4(x - 1)$

$y = 4x - 4 + 3$

$y = 4x - 1$

$n: \; y-3=-\frac{1}{4}(x-1)$

$y=-\frac{1}{4}x+\frac{1}{4}+3$

$y=-\frac{1}{4}x+\frac{13}{4}$

$4x-1=0$

$4x=1$

$x=\frac{1}{4}$

$T\left(\frac{1}{4},0\right)$

$-\frac{1}{4}x+\frac{13}{4}=0$

$\frac{1}{4}x=\frac{13}{4}$

$x=13$

$N(13,0)$

$M(1,3)$

$t=|\vec{MT}|=\sqrt{\left(\frac{1}{4}-1\right)^2+(0-3)^2}$

$t=\sqrt{\frac{9}{16}+9}=\sqrt{\frac{153}{16}}=\frac{3}{4}\sqrt{17}$

$n=|\vec{MN}|=\sqrt{(13-1)^2+(0-3)^2}$

$n=\sqrt{153}$

$s_t=1-\frac{1}{4}=\frac{3}{4}$

$s_n=13-1=12$

$P=\frac{a\,h_a}{2}$

$a=|TN|=\sqrt{\left(13-\frac{1}{4}\right)^2}=\frac{51}{4}$

$h_a=3$

$P=\frac{\frac{51}{4}\cdot 3}{2}$

$P=19{,}125$

b) $y=\frac{x^2-2x+2}{x-1}$

$M(3,y_0)$

$y'=\frac{(x^2-2x+2)'\cdot(x-1)-(x^2-2x+2)\cdot(x-1)'}{(x-1)^2}$

$y'=\frac{(2x-2)(x-1)-(x^2-2x+2)\cdot 1}{(x-1)^2}$

$y'=\frac{2x^2-2x-\cancel{2x}\cancel{+2}-x^2\cancel{+2x}\cancel{-2}}{(x-1)^2}$

$y'=\frac{x^2-2x}{(x-1)^2}$

$y'=\frac{x(x-2)}{(x-1)^2}$

$y_0=y(x_0)=\frac{9-6+2}{3-1}=\frac{5}{2}$

$y'(x_0)=\frac{3(3-2)}{(3-1)^2}=\frac{3}{4}$

$t: \; y-\frac{5}{2}=\frac{3}{4}(x-3)$

$y=\frac{3}{4}x-\frac{9}{4}+\frac{10}{4}$

$y=\frac{3}{4}x+\frac{1}{4}$

$n: \; y-\frac{5}{2}=-\frac{4}{3}(x-3)$

$y=-\frac{4}{3}x+\frac{12}{3}+\frac{5}{2}$

$y=-\frac{4}{3}x+\frac{13}{2}$

$y = \frac{3}{4}x + \frac{1}{4} = 0$

$\frac{3}{4}x = -\frac{1}{4}$

$x = \frac{-\frac{1}{4}}{\frac{3}{4}}$

$x = -\frac{1}{3}$

$T\left(-\frac{1}{3}, 0\right)$

$S_t = 3 - \left(-\frac{1}{3}\right) = \frac{10}{3}$

$y = -\frac{4}{3}x + \frac{13}{2}$

$\frac{4}{3}x = \frac{13}{2}$

$x = \frac{\frac{13}{2}}{\frac{4}{3}}$

$x = \frac{39}{8}$

$N\left(\frac{39}{8}, 0\right)$

$s_n = \frac{39}{8} - 3 = \frac{15}{8}$

$P = \frac{a \cdot h_a}{2}$

$a = |TN| = \frac{39}{8} + \frac{1}{3} = \frac{117}{24} + \frac{8}{24} = \frac{125}{24}$

$h_a = \frac{5}{2}$

$P = \frac{\frac{125}{24} \cdot \frac{5}{2}}{\frac{2}{1}}$

$P = \frac{625}{96} \approx 6,51$

- Diferencijabilna funkcija $f(x)$ je monotono rastuća (opadajuća) na nekom intervalu, ako i samo ako je $f'(x) \geq 0$ ($f'(x) \leq 0$) na tom intervalu pri čemu $f'(x) = 0$ može važiti samo u pojedinim tačkama intervala.

- Poseban uslov da diferencijabilna funkcija $f(x)$ u tački x_0 ima ekstrem (lokalni minimum ili maksimum) je da u toj tački važi $f'(x_0) = 0$. Dovoljan uslov za postojanje ekstrema u tački x_0 je da prvi izvod u tački x_0 menja znak.

- Neprekidna funkcija na zatvorenom intervalu dostiže svoju najmanju i najveću vrednost.

13. a) $y=\frac{x^4}{(1+x)^3}$

$(1+x)^3 \neq 0$

$x \neq -1$

$D = R \setminus \{-1\}$

$$y' = \frac{(x^4)' \cdot (1+x)^3 - ((1+x)^3)' \cdot x^4}{(1+x)^6}$$

$$= \frac{4x^3 \cdot (1+x)^3 - 3(1+x)^2 \cdot x^4 \cdot (1+x)'}{(1+x)^6}$$

$$= \frac{\cancel{(1+x)^2} \cdot (4x^3 \cdot (1+x) - 3x^4)}{(1+x)^{\cancel{6}\,4}}$$

$$= \frac{4x^3 + 4x^4 - 3x^4}{(1+x)^4} = \frac{4x^3 + x^4}{(1+x)^4}$$

$$= x^3 \cdot \frac{4+x}{(1+x)^4}$$

$y' = 0 \Rightarrow x^3 = 0 \quad \vee \quad x+4=0$

$x_1 = 0 \qquad x_2 = -4$

	$-\infty$ … -4	-4 … -1	-1 … 0	0 … $+\infty$
x^3	$-$	$-$	$-$	$+$
$4+x$	$-$	$+$	$+$	$+$
$(1+x)^4$	$+$	$+$	$+$	$+$
	$+$	$-$	$-$	$+$
	↗	↘	↘	↗

max: $x_2 = -4 \quad y(x_2) = -\frac{256}{27} \approx -9,48$

min: $x_1 = 0 \quad y(x_1) = 0$

b) $y = \frac{e^x}{1+x}$

$1+x \neq 0$

$x \neq -1$

$D = R \setminus \{-1\}$

$$y' = \frac{e^x \cdot (1+x) - e^x}{(1+x)^2}$$

$$y' = \frac{e^x (1+x-1)}{(1+x)^2}$$

$$y' = \frac{x e^x}{(1+x)^2}$$

$y' = 0 \Longrightarrow x = 0 \quad \vee \quad \cancel{e^x = 0}$

$x = 0$

	$-\infty$ … -1	-1 … 0	0 … $+\infty$
x	$-$	$-$	$+$
e^x	$+$	$+$	$+$
$(1+x)^2$	$+$	$+$	$+$
	$-$	$-$	$+$
	$\searrow$	$\searrow$	$\nearrow$

min: $M(0, 1)$

c) $y = \frac{\ln x}{\sqrt{x}}$

$D = \mathbb{R}^+ = (0, +\infty)$

$$y' = \frac{\frac{1}{x} \cdot \sqrt{x} - \frac{1}{2\sqrt{x}} \cdot \ln x}{x} = \frac{2 - \ln x}{2x\sqrt{x}}$$

$$= \frac{\frac{1}{\sqrt{x}} - \frac{1}{2} \ln x \cdot \frac{1}{\sqrt{x}}}{x}$$

$$= \frac{\frac{2 - \ln x}{2\sqrt{x}}}{x}$$

$y' = 0 \Rightarrow 2 - \ln x = 0$

$\ln x = 2$

$\log_e x = 2$

$e^2 = x \approx 7{,}39$

	0	e^2 $+\infty$
$2-\ln x$	$+$	$-$
$2x$	$+$	$+$
$\sqrt{x}$	$+$	$+$
	$+$	$-$
	$\nearrow$	$\searrow$

max: $x = e^2$ $\quad y(e^2) = \frac{\ln e^2}{\sqrt{e^2}} = \frac{\log_e e^2}{e} = \frac{2}{e} \approx 0{,}74$

d) $y = \sqrt{x^2 - 2}$

$x^2 - 2 \geq 0$

$x^2 \geq 2$

$D: x \in (-\infty, -2] \cup [2, +\infty)$

$y' = \frac{1}{\cancel{2}\sqrt{x^2-2}} \cdot \cancel{2}x$

$y' = \frac{x}{\sqrt{x^2-2}}$

$y' = 0 \Rightarrow x = 0 \notin R$

Nema ekstremnih vrednosti.

e) $y = \dfrac{x^2 - 2x + 2}{x - 1}$

$x - 1 \neq 0$

$x \neq 1$

$D = R \setminus \{1\}$

$y' = \dfrac{(2x-2)(x-1) - x^2 + 2x - 2}{(x-1)^2}$

$y' = \dfrac{2x^2 - 2x - \cancel{2x} + \cancel{2} - x^2 + \cancel{2x} - \cancel{2}}{(x-1)^2}$

$y' = \dfrac{x^2 - 2x}{(x-1)^2}$

$y' = 0 \Rightarrow x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0 \vee x = 2$

	$-\infty$ … 0	0 … 1	1 … 2	2 … $+\infty$
x	−	+	+	+
$(x-2)$	−	−	−	+
$(x-1)^2$	+	+	+	+
	+	−	−	+
	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

max: $x = 0$, $y = -2$

min: $x = 2$, $y = 2$

12.5 Zadaci za vežbu

a) $y^2 - x^2 = 2$

$y^2 = x^2 + 2$

$y_1 = \sqrt{x^2+2}$ $\quad y_2 = -\sqrt{x^2+2}$

$D: x^2+2 \geq 0$ $\quad K: y_1 \in [\sqrt{2}, +\infty)$

$x^2 \geq -2$ $\quad y_1 = \sqrt{x^2+2} : R \to [\sqrt{2}, +\infty)$

$D \in R$

$D: x^2+2 \geq 0$ $\quad K: y_2 \in (-\infty, -\sqrt{2}]$

$x^2 \geq -2$ $\quad y_2 = -\sqrt{x^2+2} : R \to (-\infty, -\sqrt{2}]$

$D \in R$

$\left(\frac{x-1}{2}\right)^2 + y^2 = 4$

$y^2 = 4 - \left(\frac{x-1}{2}\right)^2$

$y^2 = 4 - \left(\frac{1}{2}\right)^2 \cdot (x-1)^2$

$y_1 = \sqrt{4 - \frac{1}{4}(x-1)^2}$ $\quad y_2 = -\sqrt{4 - \frac{1}{4}(x-1)^2}$

$D: 4 - \frac{1}{4}(x-1)^2 \geq 0$ $\quad x_{1,2} = \frac{2 \pm \overset{8}{\sqrt{4+60}}}{2} \to x_1 = 5, \; x_2 = -3$

$-\frac{x^2-2x+1}{4} \geq -4$

$x \in [-3, 5]$

$\frac{x^2-2x+1}{4} \leq 4$

$x^2 - 2x + 1 \leq 16$

$x^2 - 2x - 15 \leq 0$

$K: y_1 \in [0,2]$

$y_1 = \sqrt{4-\frac{1}{4}(x-1)^2} : [-3,5] \to [0,2]$

$K: y_2 \in [-2,0]$

$y_2 = -\sqrt{4-\frac{1}{4}(x-1)^2} : [-3,5] \to [-2,0]$

c) $x^2 - y = 2$ $\qquad K: y \in [-2,+\infty)$

$y = x^2 - 2$ $\qquad y = x^2-2 : R \to [-2,+\infty)$

$D \in R$

2. a) $a_n = \frac{1}{2n} + \frac{2n}{3n+1}$

$$\lim_{n\to\infty} \frac{1}{2n} + \frac{2n}{3n+1} = \lim_{n\to\infty} \frac{3n+1+4n^2}{6n^2+2n} = \lim_{n\to\infty} \frac{\cancel{n^2}(4+\frac{3}{n}\to 0+\frac{1}{n^2}\to 0)}{\cancel{n^2}(6+\frac{2}{n}\to 0)} = \frac{4}{6} = \frac{2}{3}$$

b) $a_n = \frac{n^2+n-1}{(n+1)^2}$

$$\lim_{n\to\infty} \frac{n^2+n-1}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2+n-1}{n^2+2n+1} = \lim_{n\to\infty} \frac{\cancel{n^2}(1+\frac{1}{n}\to 0-\frac{1}{n^2}\to 0)}{\cancel{n^2}(1+\frac{2}{n}\to 0+\frac{1}{n^2}\to 0)} = 1$$

c) $a_n = \frac{3(n+1)(n+2)(n-1)}{n^3-2n^2+1} = \frac{3(n^2+2n+n+2)(n-1)}{n^3-2n^2+1} = \frac{3(n^3-\cancel{n^2}+2n^2-\cancel{2n}+\cancel{n^2}-n+\cancel{2n}-2)}{n^3-2n^2+1}$

$= \frac{3n^3+6n^2-3n-6}{n^3-2n^2+1}$

$$\lim_{n\to\infty} \frac{3n^3+6n^2-3n-6}{n^3-2n^2+1} = \lim_{n\to\infty} \frac{\cancel{n^3}(3+\frac{6}{n}\to 0-\frac{3}{n^2}\to 0-\frac{6}{n^3}\to 0)}{\cancel{n^3}(1-\frac{2}{n}\to 0+\frac{1}{n^3}\to 0)} = 3$$

d) $a_n = \frac{\sin(3n)}{n}$

$\lim_{n\to\infty} \frac{\sin(3n)}{n} = \frac{0}{n} = 0$

e) $a_n = \frac{1}{n}\cos n^2$

$\lim_{n\to\infty} \frac{1}{n}\cos n^2 = 0$

f) $a_n = \frac{2n}{3n+1}\cdot\frac{n}{1-3n} = \frac{2n^2}{\cancel{3n}-9n^2+1-\cancel{3n}} = \frac{2n^2}{-9n^2+1}$

$\lim_{n\to\infty} \frac{2n^2}{-9n^2+1} = \lim_{n\to\infty} \frac{\cancel{n^2}\cdot 2}{\cancel{n^2}\left(-9+\frac{1}{n^2}\right)} = -\frac{2}{9}$

g) $a_n = \frac{n-3^n}{2^n+3^{n+1}}$

$\lim_{n\to\infty} \frac{n-3^n}{2^n+3^{n+1}} = \frac{\cancel{3^n}\cdot\left(\frac{n}{3^n}-1\right)}{\cancel{3^n}\left(3+\left(\frac{2}{3}\right)^n\right)} = -\frac{1}{3}$

i) $a_n = \frac{4^n-5^n}{\sqrt[3]{n^2}+2^n}$

$\lim_{n\to\infty} \frac{4^n-5^n}{n^{\frac{2}{3}}+2^n} = \lim_{n\to\infty} \frac{\cancel{5^n}\left(\left(\frac{4}{5}\right)^n-1\right)}{\cancel{5^n}\left(\frac{n^{\frac{2}{3}}}{5^n}+\left(\frac{2}{5}\right)^n\right)} = \frac{-1}{0} = -\infty$

) $a_n = \frac{2\cdot 4^n+3\cdot 5^n}{2\cdot 3^n-3\cdot 5^n}$

$$\lim_{n\to\infty} \frac{2\cdot 4^n + 3\cdot 5^n}{2\cdot 3^n - 3\cdot 5^n} = \lim_{n\to\infty} \frac{\cancel{5^n}\left(2\left(\frac{4}{5}\right)^n + 3\right)}{\cancel{5^n}\left(2\cdot\left(\frac{3}{5}\right)^n - 3\right)} = -1$$

j) $a_n = \frac{(-2)^n + 5^n}{(-2)^{n+1} + 5^{n+1}}$

$$\lim_{n\to\infty} \frac{(-2)^n + 5^n}{(-2)^{n+1} + 5^{n+1}} = \lim_{n\to\infty} \frac{\cancel{5^n}\left(\left(-\frac{2}{5}\right)^n + 1\right)}{\cancel{5^n}\left((-2)\cdot\left(-\frac{2}{5}\right)^n + 5\right)} = \frac{1}{5}$$

k) $a_n = \sqrt{n+3} - \sqrt{n}$

$$\lim_{n\to\infty} \sqrt{n+3} - \sqrt{n} \cdot \frac{\sqrt{n+3} + \sqrt{n}}{\sqrt{n+3} + \sqrt{n}} = \lim_{n\to\infty} \frac{\cancel{n} + 3 - \cancel{n}}{\sqrt{n+3} + \sqrt{n}} = \lim_{n\to\infty} \frac{3}{\infty} = 0$$

l) $a_n = \sqrt{n}\left(\sqrt{n+3} - \sqrt{n}\right) = \sqrt{n^2+3n} - n$

$$\lim_{n\to\infty} \sqrt{n^2+3n} - n \cdot \frac{\sqrt{n^2+3n} + n}{\sqrt{n^2+3n} + n} = \lim_{n\to\infty} \frac{\cancel{n^2} + 3n - \cancel{n^2}}{\sqrt{n^2+3n} + n} = \lim_{n\to\infty} \frac{3\cancel{n}}{\cancel{n}\left(\sqrt{1+\frac{3}{n}} + 1\right)} = \frac{3}{2}$$

m) $$\lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \frac{\cancel{n} + 1 - \cancel{n}}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \frac{1}{\infty} = 0$$

n) $$\lim_{n\to\infty} \left(\sqrt{n^2+n} - \sqrt{n^2+1}\right) \cdot \frac{\sqrt{n^2+n} + \sqrt{n^2+1}}{\sqrt{n^2+n} + \sqrt{n^2+1}} = \lim_{n\to\infty} \frac{\cancel{n^2} + n - \cancel{n^2} - 1}{\sqrt{n^2+n} + \sqrt{n^2+1}} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\left(1 - \frac{1}{n}\right)}{\cancel{n}\left(\sqrt{1+\frac{1}{n}} + \sqrt{1+\frac{1}{n^2}}\right)} = \frac{1}{2}$$

3\. a) $y = \dfrac{x^3+7}{x^2-5x+6}$

$x^2-5x+6 \neq 0,$

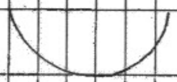
$x_{1,2} \neq \dfrac{5 \pm \sqrt{25-24}}{2}$ → $x_1 \neq 3$, $x_2 \neq 2$

$D: \mathbb{R} \setminus \{2, 3\}$

b) $y = \sqrt{2x^2+x+9}$

$2x^2+x+9 \geq 0$

$x_{1,2} = \dfrac{-1 \pm \sqrt{1-72}}{4}$



\+ \+ \+ \+ \+

$D: \mathbb{R}$

c) $y = \sqrt{x+1} - \sqrt{3-x} + e^{\frac{1}{x}}$

$x+1 \geq 0$ $\quad$ $3-x \geq 0$ $\quad$ $x \neq 0$

$x \geq -1$ $\quad$ $x \leq 3$

$D: (-1, 0) \cup (0, 3)$

d) $y = \dfrac{x^2-4-\ln(-x)}{1+\sqrt{x^2-4}}$ $\quad$ $x^2-4 \geq 0$ $\quad \wedge \quad$ $\log_e(-x)$

$x^2 \geq 4$ $\quad$ $-x > 0$

$x \in (-\infty, -2] \cup [2, +\infty)$ $\quad$ $x < 0$

$D: (-\infty, -2]$

e) $y=\frac{\sqrt{x+1}}{\ln(1-x)}$ $\qquad x+1\geq 0 \qquad \log_e(1-x)$

$x\geq -1 \qquad 1-x>0$

$\ln(1-x)\neq 0 \qquad -x>-1$

$1-x\neq 1 \qquad x<1$

$x\neq 0 \qquad D: [-1,0)\cup(0,1)$

f) $y=\ln(x-|x|)$

$x-|x|>0$

I $x>0 \quad x-x>0$ $\qquad$ II $x<0 \quad x-(-x)>0$

$0>0 \quad \perp$ $\qquad 2x>0$

$x>0$ - ne ispunjava uslov

$D: \emptyset$

4. a) $\lim\limits_{x\to\infty}\frac{(x-1)^3}{2x^3-x+2}=\lim\limits_{x\to\infty}\frac{x^3-3x^2+3x-1}{2x^3-x+2}=\lim\limits_{x\to\infty}\frac{\cancel{x^3}\left(1-\frac{3}{x}^{\to 0}+\frac{3}{x^2}^{\to 0}-\frac{1}{x^3}^{\to 0}\right)}{\cancel{x^3}\left(2-\frac{1}{x^2}^{\to 0}+\frac{2}{x^3}^{\to 0}\right)}=\frac{1}{2}$

b) $\lim\limits_{x\to\infty}\left(1+\frac{1}{\sqrt{x+1}}\right)=\lim\limits_{x\to\infty}\frac{\sqrt{x+1}+1}{\sqrt{x+1}}=\lim\limits_{x\to\infty}\frac{\cancel{\sqrt{x}}\left(\sqrt{1+\frac{1}{x}^{\to 0}}+\frac{1}{\sqrt{x}}^{\to 0}\right)}{\cancel{\sqrt{x}}\sqrt{1+\frac{1}{x}^{\to 0}}}=1$

c) $\lim\limits_{x\to\infty}\frac{2x^2-3x+4}{\sqrt{x^4+1}}=\lim\limits_{x\to\infty}\frac{\cancel{x^2}\left(2-\frac{3}{x}^{\to 0}+\frac{4}{x^2}^{\to 0}\right)}{\cancel{x^2}\left(\sqrt{1+\frac{1}{x^4}^{\to 0}}\right)}=2$

d) $\lim\limits_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x}}}=\lim\limits_{x\to\infty}\frac{\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}\left(1+\sqrt{\frac{1}{x}}^{\to 0}\right)}=1$

e) $\lim\limits_{x\to 1}\frac{2x^2-x-1}{x^2-1}=\lim\limits_{x\to 1}\frac{(2x+1)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}}=\frac{3}{2}$

$(2x^2-x-1):(x-1)=2x+1$
$\underline{-2x^2+2x}$
$\quad x-1$
$\underline{\;-x+1}$
$\quad //$

$$\lim_{x\to 3}\frac{x^2-5x+6}{x^2-8x+15}=\lim_{x\to 3}\frac{\cancel{(x-3)}(x-2)}{\cancel{(x-3)}(x-5)}=-\frac{1}{2}$$

$$\begin{array}{l}(x^2-5x+6):(x-3)=x-2\\ \underline{\cancel{x^2}+3x}\\ \quad -2x+6\\ \quad \underline{2x-6}\\ \quad //\end{array}\qquad\qquad \begin{array}{l}(x^2-8x+15):(x-3)=x-5\\ \underline{\cancel{-x^2}+3x}\\ \quad -5x+15\\ \quad \underline{5x-15}\\ \quad //\end{array}$$

$$\lim_{x\to 1}\left(\frac{3}{1-x^3}-\frac{1}{1-x}\right)=\lim_{x\to 1}\left(\frac{3}{(1-x)(1+x+x^2)}-\frac{1}{1-x}\right)=$$

$$\lim_{x\to 1}\left(\frac{3-1-x-x^2}{(1-x)(1+x+x^2)}\right)=\lim_{x\to 1}\frac{-x^2-x+2}{(1-x)(1+x+x^2)}=\lim_{x\to 1}\frac{(-x-2)\cancel{(x-1)}}{\cancel{(x-1)}(-x^2-x-1)}$$

$$\begin{array}{l}(-x^2-x+2):(x-1)=-x-2\\ \underline{\cancel{x^2}-x}\\ \quad -2x+2\\ \quad \underline{2x-2}\\ \quad //\end{array}\qquad\qquad =\frac{-3}{-3}=1$$

$$\lim_{x\to -1}\frac{x^3+x^2-x-1}{x^3+2x^2-x-2}=\lim_{x\to -1}\frac{x^2(x+1)-(x+1)}{x^2(x+2)-(x+2)}=\lim_{x\to -1}\frac{\cancel{(x^2-1)}(x+1)}{\cancel{(x^2-1)}(x+2)}=\frac{0}{1}=0$$

$$\lim_{x\to 4}\frac{x\sqrt{x}-4\sqrt{x}}{x^2-16}=\lim_{x\to 4}\frac{\sqrt{x}\,\cancel{(x-4)}}{\cancel{(x-4)}(x+4)}=\frac{2}{8}=\frac{1}{4}$$

$$\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2}=\lim_{x\to 2}\frac{\cancel{\sqrt{x}-\sqrt{2}}}{\cancel{(\sqrt{x}-\sqrt{2})}(\sqrt{x}+\sqrt{2})}=\frac{1}{2\sqrt{2}}$$

$$\lim_{x\to 0}\frac{x}{\sqrt{1+3x}-1}=\lim_{x\to 0}\frac{\frac{\cancel{(\sqrt{1+3x}-1)}(\sqrt{1+3x}+1)}{3}}{\cancel{\sqrt{1+3x}-1}}=\lim_{x\to 0}\frac{\sqrt{1+3x}+1}{3}=\frac{2}{3}$$

$$\lim_{x\to 6}\frac{\sqrt[3]{4x+3}-3}{2x-12}\cdot\frac{(\sqrt[3]{4x+3})^2+3\sqrt[3]{4x+3}+9}{(\sqrt[3]{4x+3})^2+3\sqrt[3]{4x+3}+9}=$$

$$\lim_{x\to 6}\frac{4x+3-27}{(2x-12)\left((\sqrt[3]{4x+3})^2+3\sqrt[3]{4x+3}+9\right)}=\lim_{x\to 6}\frac{4x-24}{(2x-12)\left((\sqrt[3]{4x+3})^2+3\sqrt[3]{4x+3}+9\right)}=$$

$$= \lim_{x\to 6} \frac{\overset{2}{\cancel{4}}\cancel{(x-6)}}{\cancel{2}\cancel{(x-6)}\left((\sqrt[3]{4x+3})^2+3\sqrt[3]{4x+3}+9\right)} = \frac{2}{9+3\cdot 3+9} = \frac{2}{27}$$

m) $$\lim_{x\to 2} \frac{\sqrt{x+2}-2}{\sqrt{x}-\sqrt{2}} \cdot \frac{\sqrt{2+x}+2}{\sqrt{2+x}+2} = \lim_{x\to 2} \frac{x-2}{(\sqrt{x}-\sqrt{2})(\sqrt{2+x}+2)} \cdot \frac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+\sqrt{2}} =$$

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}(\sqrt{x}+\sqrt{2})}{\cancel{(x-2)}(\sqrt{x+2}+2)} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$$

n) $$\lim_{x\to 4} \frac{\sqrt[3]{x}-\sqrt[3]{4}}{x-4} \cdot \frac{(\sqrt[3]{x})^2+\sqrt[3]{4x}+(\sqrt[3]{4})^2}{(\sqrt[3]{x})^2+\sqrt[3]{4x}+(\sqrt[3]{4})^2} = \lim_{x\to 4} \frac{\cancel{x-4}}{\cancel{(x-4)}\left((\sqrt[3]{x})^2+\sqrt[3]{4x}+(\sqrt[3]{4})^2\right)} =$$

$$= \frac{1}{\sqrt[3]{16}+\sqrt[3]{16}+\sqrt[3]{16}} = \frac{1}{3\sqrt[3]{16}}$$

o) $$\lim_{x\to 4} \frac{\sqrt{1+2x}-3}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4} \frac{(\sqrt{1+2x}-3)(\sqrt{x}+2)}{x-4} \cdot \frac{\sqrt{1+2x}+3}{\sqrt{1+2x}+3} =$$

$$= \lim_{x\to 4} \frac{(1+2x-9)(\sqrt{x}+2)}{(x-4)(\sqrt{1+2x}+3)} = \lim_{x\to 4} \frac{2\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{(x-4)}(\sqrt{1+2x}+3)} = \frac{2\cdot 4}{3+3} = \frac{8}{6} = \frac{4}{3}$$

5. a) $$\lim_{x\to\infty} \left(\sqrt{x^2+3x}-x\right) \cdot \frac{\sqrt{x^2+3x}+x}{\sqrt{x^2+3x}+x} = \lim_{x\to\infty} \frac{\cancel{x^2}+3x-\cancel{x^2}}{\sqrt{x^2+3x}+x} =$$

$$= \lim_{x\to\infty} \frac{3\cancel{x}}{\cancel{x}\left(\sqrt{1+\frac{3}{x}^{\,0}}+1\right)} = \frac{3}{2}$$

b) $$\lim_{x\to\infty} \left(\sqrt{x^2-5x+4}-x\right) \cdot \frac{\sqrt{x^2-5x+4}+x}{\sqrt{x^2-5x+4}+x} = \lim_{x\to\infty} \frac{\cancel{x^2}-5x+4-\cancel{x^2}}{\sqrt{x^2-5x+4}+x} =$$

$$= \lim_{x\to\infty} \frac{\cancel{x}\left(-5+\frac{4}{x}^{\,0}\right)}{\cancel{x}\left(\sqrt{1-\frac{5}{x}^{\,0}+\frac{4}{x^2}^{\,0}}+1\right)} = \frac{-5}{2}$$

c) $\lim_{x\to\infty}\left(\sqrt{x^2+1}-\sqrt{x^2-4x}\right)\cdot\frac{\sqrt{x^2+1}+\sqrt{x^2-4x}}{\sqrt{x^2+1}+\sqrt{x^2-4x}}=$

$=\lim_{x\to\infty}\frac{\cancel{x^2}+1-\cancel{x^2}+4x}{\sqrt{x^2+1}+\sqrt{x^2-4x}}=\lim_{x\to\infty}\frac{\cancel{x}\left(4+\frac{1}{x}\right)}{\cancel{x}\left(\sqrt{1+\frac{1}{x^2}}+\sqrt{1-\frac{4}{x}}\right)}=2$

6. a) $y=\frac{x}{1-x^2}$

$y'=\frac{1\cdot(1-x^2)-(-2x)\cdot x}{(1-x^2)^2}=\frac{1-x^2+2x^2}{(1-x^2)^2}=\frac{x^2+1}{(1-x^2)^2}$

b) $y=\frac{\sqrt{x}}{\sqrt{x}+1}$

$y'=\frac{\frac{1}{2\sqrt{x}}\cdot(\sqrt{x}+1)-\frac{1}{2\sqrt{x}}\cdot\sqrt{x}}{(\sqrt{x}+1)^2}=\frac{\frac{\sqrt{x}+1}{2\sqrt{x}}-\frac{\sqrt{x}}{2\sqrt{x}}}{(\sqrt{x}-1)^2}=\frac{\frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2}=\frac{1}{2\sqrt{x}\,(\sqrt{x}+1)^2}$

c) $y=(x+1)(x-1)$

$y'=1(x-1)+1(x+1)=x-\cancel{1}+x+\cancel{1}=2x$

d) $y=\frac{\sin^2 x}{\cos x}$

$y'=\frac{2\sin x\cdot\cos x\cdot\cos x+\sin x\cdot\sin^2 x}{\cos^2 x}=\frac{2\sin x\cos^2 x+\sin^3 x}{\cos^2 x}=$

$=\frac{\sin x\,(2\cos^2 x+\sin^2 x)}{\cos^2 x}$

e) $y=x\arcsin x$

$y'=1\cdot\arcsin x+\frac{x}{\sqrt{1-x^2}}=\arcsin x+\frac{x}{\sqrt{1-x^2}}$

f) $y = \operatorname{arctg} x - \frac{x}{1+x^2}$

$$y' = \frac{1}{1+x^2} - \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1+x^2}{(1+x^2)^2} - \frac{-x^2+1}{(1+x^2)^2} = \frac{\cancel{1}+x^2+x^2-\cancel{1}}{(1+x^2)^2} = \frac{2x^2}{(1+x^2)^2}$$

g) $y = xe^{-x}$

$$y' = e^{-x} + x\left(\frac{1}{e^x}\right)' = e^{-x} + x \cdot \frac{-e^x}{e^{2x}} = e^{-x} - xe^{-x} = e^{-x}(1-x)$$

h) $y = 2^x + x^2 = 2^x \ln 2 + 2x$

i) $y = \frac{1-\ln x}{1+\ln x} = \frac{-\frac{1}{x}(1+\ln x) - \frac{1}{x}(1-\ln x)}{(1+\ln x)^2} = \frac{-\frac{1}{x}(1+\cancel{\ln x}+1-\cancel{\ln x})}{(1+\ln x)^2} =$

$$= \frac{-\frac{2}{x}}{(1+\ln x)^2} = \frac{-2}{x(1+\ln x)^2}$$

j) $y = \sin(3x)$

$y' = \cos 3x \cdot 3 = 3\cos 3x$

k) $y = \operatorname{arctg} \frac{1+x}{1-x}$

$$y' = \frac{1}{1+\left(\frac{1+x}{1-x}\right)^2}\left(\frac{1+x}{1-x}\right)' = \frac{1}{1+\frac{1+2x+x^2}{1-2x+x^2}} \cdot \frac{1-\cancel{x}+1+\cancel{x}}{(1-x)^2} =$$

$$= \frac{1}{\frac{1-\cancel{2x}+x^2+1+\cancel{2x}+x^2}{1-2x+x^2}} \cdot \frac{2}{(1-x)^2} = \frac{\cancel{(1-x)^2}}{\cancel{2}(x^2+1)} \cdot \frac{\cancel{2}}{\cancel{(1-x)^2}} = \frac{1}{x^2+1}$$

l) $y = e^{x^2-x+2}$

$y' = e^{x^2-x+2} \cdot (2x-1) = (2x-1)e^{x^2-x+2}$

m) $y = \log_2(1 - x\cos x)$

$$y' = \frac{1}{(1-x\cos x)\ln 2} \cdot (-1)\cos x + (-\sin x)\cdot(-x)$$

$$= \frac{1}{(1-x\cos x)\ln_2} \cdot x\sin x - \cos x = \frac{x\sin x - \cos x}{(1-x\cos x)\ln 2}$$

7. a) $y = x^5 - 5x^4 + 5x^3 + 1$ za $x \in [-1, 2]$

$y' = 5x^4 - 20x^3 + 15x^2$

$y' = 0 \Rightarrow 5x^4 - 20x^3 + 15x^2 = 0$

$5x^2(x^2 - 4x + 3) = 0$

$5x^2(x-3)(x-1) = 0$

$x = 0 \quad \vee \quad \cancel{x = 3} \quad \vee \quad x = 1$

$x \in [-1, 2]$

x	(-1, 0)	(0, 1)	(1, 2)
y'	+	+	−
y	↗	↗	↘

$x = 1 \rightarrow y = 2 \quad \max(1, 2)$

$$\left.\begin{array}{l} x = -1 \rightarrow y = -10 \\ x = 2 \rightarrow y = -7 \end{array}\right\} \min(-1, -10)$$

b) $y = \frac{1}{x^2} + \frac{1}{(1-x^2)^2}$

$$y' = (x^{-2})' + \left(\frac{1}{(1-x^2)^2}\right)' =$$

$$= -2x^{-3} + \frac{0 - 2(1-x^2)(1-x^2)'}{(1-x^2)^4} =$$

$$= -\frac{2}{x^3} + \frac{-2(1-x^2)(-2x)}{(1-x^2)^4} = -\frac{2}{x^3} + \frac{4x\cancel{(1-x^2)}}{(1-x^2)^{\cancel{4}\,3}} =$$

$$= -\frac{2}{x^3} + \frac{4x}{(1-x^2)^3} = \frac{4x^4 - 2(1-x^2)^3}{x^3(1-x^2)^3} = \frac{4x^4 - 2(1-3x^2+3x^4-x^6)}{x^3(1-x^2)^3} =$$

$$= \frac{4x^4 - 2 + 6x^2 - 6x^4 + 2x^6}{x^3(1-x^2)^3} = \frac{2x^6 - 2x^4 + 6x^2 - 2}{x^3(1-x^2)^3}$$

* OKO OVOG ZADATKA SMO SE I PROFESOR I JA PATILI JER SMO OBOJE DOŠLI DO REŠENJA DA y' NEMA REALNIH NULA. AKO UOČITE GREŠKU NEGDE, ISPRAVITE ME.

c) $y = \frac{x}{x^2+2x+4}$

$$y' = \frac{x^2+2x+4-(2x+2)\cdot x}{(x^2+2x+4)^2}$$

$$y' = \frac{x^2+\cancel{2x}+4-2x^2-\cancel{2x}}{(x^2+2x+4)^2}$$

$$y' = \frac{-x^2+4}{(x^2+2x+4)^2} = \frac{(2-x)\cdot(2+x)}{(x^2+2x+4)^2}$$

$y' = 0 \Rightarrow 2-x=0 \quad \vee \quad 2+x=0$

$x = 2 \qquad x = -2$

	$-\infty$	-2	2	$+\infty$
$2-x$	$+$	$+$	$-$	
$2+x$	$-$	$+$	$+$	
$(x^2+2x+4)^2$	$+$	$+$	$+$	
	$-$	$+$	$-$	
	↘ min	↗ max	↘	

min: $x = -2$, $y = -\frac{1}{2}$

max: $x = 2$, $y = \frac{1}{6}$

1) $y = \frac{5-x}{9-x^2}$, za $x \in [0,2]$

$y' = \frac{-1(9-x^2)+2x(5-x)}{(9-x^2)^2}$

$y' = \frac{-9+x^2+10x-2x^2}{(9-x^2)^2}$

$y' = \frac{-x^2+10x-9}{(9-x^2)^2}$

$y' = 0 \Rightarrow -x^2+10x-9=0$

$x_{1,2} = \frac{-10 \pm \sqrt{100-36}}{-2}$ $\nearrow x_1 = 1$, $\searrow \cancel{x_2 = 9}$ $x \in [0,2]$

x	$(0,1)$	$(1,2)$
y'	$-$	$+$
y	$\searrow$	$\nearrow$

$x = 1 \rightarrow y = \frac{1}{2}$ min $(1, \frac{1}{2})$

$$\left.\begin{array}{l} x = 0 \rightarrow y = \frac{5}{9} \\ x = 2 \rightarrow y = \frac{3}{5} \end{array}\right\} \text{max } \left(2, \frac{3}{5}\right)$$

$f(x) = \frac{x^3-4}{x^2}$ $M(2, y_0)$

$y' = \frac{3x^4 - 2x(x^3-4)}{x^4} = \frac{3x^4 - 2x^4 + 8x}{x^4} = \frac{x^4+8x}{x^4} = \frac{\cancel{x}(x^3+8)}{x^{\cancel{4}\,3}} = \frac{x^3+8}{x^3}$

$x_0 = 2 \Rightarrow y_0 = \frac{8-4}{4} = \frac{4}{4} = 1 \qquad M(2,1)$

$y_0' = \frac{8+8}{8} = \frac{16}{8} = 2$

$t: y - 1 = 2(x-2)$

$y - 1 = 2x - 4$

$y = 2x - 3$

$T: y = 2x - 3 = 0$

$2x = 3$

$x = \frac{3}{2}$

$T\left(\frac{3}{2}, 0\right)$

$t = |\overrightarrow{MT}| = \sqrt{\frac{1}{4} + 1} = \frac{\sqrt{5}}{2}$

$st = 2 - \frac{3}{2} = \frac{1}{2}$

b) $f(x) = \frac{x^2+2}{2x-3} \qquad M(1, y_0)$

$y' = \frac{2x(2x-3) - 2(x^2+2)}{(2x-3)^2} = \frac{4x^2 - 6x - 2x^2 - 4}{(2x-3)^2} = \frac{2x^2 - 6x - 4}{(2x-3)^2}$

$x_0 = 1 \Rightarrow y_0 = \frac{1+2}{2-3} = -3$

$y_0' = \frac{2-6-4}{(-1)^2} = -8$

$t: y + 3 = -8(x-1)$

$y + 3 = -8x + 8$

$y = -8x + 5$

$T: y = -8x + 5 = 0$

$8x = 5$

$x = \frac{5}{8}$

$T\left(\frac{5}{8}, 0\right)$

$t = \sqrt{\frac{9}{64} + 9}$

$t = \frac{\sqrt{585}}{8}$

$st = 1 - \frac{5}{8} = \frac{3}{8}$

c) $f(x) = \ln \frac{2x^2+3}{x^2+1} \qquad M(1, y_0)$

$y' = \frac{x^2+1}{2x^2+3} \cdot \frac{4x(x^2+1) - 2x(2x^2+3)}{(x^2+1)^2} = \frac{4x^3 + 4x - 4x^3 - 6x}{(2x^2+3)(x^2+1)} = \frac{-2x}{(2x^2+3)(x^2+1)}$

$x_0 = 1 \Rightarrow y_0 = \ln\frac{2+3}{1+1} = \ln\frac{5}{2}$

$y_0' = \frac{-2}{(2+3)(1+1)} = \frac{-2}{10} = -\frac{1}{5}$

$t: y - \ln\frac{5}{2} = -\frac{1}{5}(x-1)$

$y - \ln\frac{5}{2} = -\frac{1}{5}x + \frac{1}{5}$

$y = -\frac{1}{5}x + \frac{1}{5} + \ln\frac{5}{2}$

$T: y = -\frac{1}{5}x + \frac{1}{5} + \ln\frac{5}{2} = 0$

$\frac{1}{5}x = \frac{1}{5} + \ln\frac{5}{2} \quad | \cdot 5$

$x = 1 + 5\ln\frac{5}{2}$

$T\left(1 + 5\ln\frac{5}{2}, 0\right)$

$t = \sqrt{\left(5\ln\frac{5}{2}\right)^2 + \left(-\ln\frac{5}{2}\right)^2}$

$St = 1 + 5\ln\frac{5}{2} - 1 = 5\ln\frac{5}{2}$

$t = \sqrt{25\ln^2\frac{5}{2} + \ln^2\frac{5}{2}}$

$t = \sqrt{26}\ln\frac{5}{2}$

$y = 4x^2 \quad M(2, y_0)$

$y' = 8x$

$x_0 = 2 \Rightarrow y_0 = 16$

$y_0' = 16$

$t: y - 16 = 16(x-2)$

$y - 16 = 16x - 32$

$y = 16x - 16$

$n: y - 16 = -\frac{1}{16}(x-2)$

$y - 16 = -\frac{1}{16}x + \frac{2}{16}$

$y = -\frac{x}{16} + \frac{129}{8}$

$T: \; 16x - 16 = 0$ $\qquad$ $N: -\frac{x}{16} + \frac{129}{8} = 0$

$16x = 16$ $\qquad$ $-\frac{x}{16} = -\frac{258}{16}$

$x = 1$ $\qquad$ $x = 258$

$T(1,0)$ $\qquad$ $U(258,0)$

$st = 2 - 1 = 1$ $\qquad$ $su = 258 - 2 = 256$

$P = \frac{16 \cdot 257}{2} = 2056$

10. a) $f(x) = \frac{e^x}{x-1}$ $\quad$ $M(0, y_0)$

$f'(x) = \frac{e^x(x-1) - e^x}{(x-1)^2} = \frac{e^x(x-2)}{(x-1)^2}$

$x_0 = 0 \Rightarrow y_0 = \frac{1}{-1} = -1$ $\qquad$ $n: \; y + 1 = \frac{1}{2}(x - 0)$ $\qquad$ $N: \; \frac{1}{2}x - 1 = 0$

$y + 1 = \frac{1}{2}x$ $\qquad$ $\frac{1}{2}x = 1$

$y_0' = \frac{-2}{1} = -2$ $\qquad$ $y = \frac{1}{2}x - 1$ $\qquad$ $x = 2$

$N(2,0)$

$n = \sqrt{4 + 1} = \sqrt{5}$ $\qquad$ $su = 2 - 0 = 2$

b) $f(x) = \frac{x^2 + 2x - 1}{x^2}$ $\quad$ $M(2, y_0)$

$f'(x) = \frac{(2x+2)x^2 - 2x(x^2+2x-1)}{x^4} = \frac{\cancel{2x^3} + 2x^2 - \cancel{2x^3} - 4x^2 + 2x}{x^4} = \frac{2x - 2x^2}{x^4}$

$x_0 = 2 \Rightarrow y_0 = \frac{7}{4}$

$y_0' = -\frac{1}{4}$

$t: \ y - \frac{7}{4} = 4(x-2)$

$y - \frac{7}{4} = 4x - 8$

$y = 4x - \frac{25}{4}$

$N: \ 4x - \frac{25}{4} = 0$

$4x = \frac{25}{4}$

$x = \frac{25}{16}$

$N\left(\frac{25}{16}, 0\right)$

$t = \sqrt{\frac{49}{256} + \frac{49}{16}}$

$t = \sqrt{\frac{833}{256}}$

$t = \frac{7\sqrt{17}}{16}$

$S_t = \frac{32}{16} - \frac{25}{16} = \frac{7}{16}$

c) $f(x) = (x^2-3)e^{-x} \quad M(0, y_0)$

$f'(x) = 2x \cdot e^{-x} - e^{-x}(x^2-3)$
$= e^{-x}(2x - x^2 + 3)$

$x_0 = 0 \Rightarrow y_0 = -3$

$y_0' = 3$

$t: \ y + 3 = -\frac{1}{3}(x-0)$

$y + 3 = -\frac{1}{3}x$

$y = -\frac{1}{3}x - 3$

$N: \ -\frac{1}{3}x - 3 = 0$

$-\frac{1}{3}x = 3$

$x = -9$

$N(-9, 0)$

$t = \sqrt{81 + 9} = \sqrt{90} = 3\sqrt{10}$

$S_t = 0 + 9 = 9$