9. Progresije

9.1. Aritmetička progresija

- Niz $a_1, a_2, \dots, a_n, \dots$ je aritmetički sa diferencijom (razlikom) d ako j
$a_2 - a_1 = a_3 - a_2 = a_n + a_{n-1} = \dots = d$
- Uobičajeno je da se za skup konačno mnogo uzastopnih članova aritmetičkog niza koristi termin aritmetička progresija.
- Za n-ti član niza važi $a_n = a_1 + (n-1)d$

$$a_n = \frac{a_{n-1} + a_{n+1}}{2} = \frac{a_{n-j} + a_{n+j}}{2}, \quad j = 2, \dots, n-1$$

- Zbir prvih n članova aritmetičkog niza je

$$S_n = n \cdot \frac{a_1 + a_n}{2} = \frac{n}{2}\left(2a_1 + (n-1)d\right)$$

- Ako između brojeva a i b treba interpolirati (umetnuti) k brojeva tako da zajedno sa a i b čine aritmetičku progresiju, onda je diferencija te progresije $d = \frac{b-a}{k+1}$

1. $a_2 + a_5 - a_3 = 10$
$a_2 + a_9 = 17$

$a_1 + d + \cancel{a_1} + 4d - \cancel{a_1} - 2d = 10$
$a_1 + 3d = 10$
$a_1 + d + a_1 + 8d = 17$
$2a_1 + 9d = 17$

$a_2 = a_1 + d$
$a_3 = a_1 + 2d$
$a_5 = a_1 + 4d$
$a_9 = a_1 + 8d$

$a_1 + 3d = 10 \; | \cdot (-2)$
$2a_1 + 9d = 17$

$-\cancel{2a_1} - 6d = -20$
$\cancel{2a_1} + 9d = 17$

$3d = -3$
$d = -1$
$a_1 = 13$

2. $S_n = 7n^2 + 5n$

I $n = 1$

$S_1 = a_1 = 7 + 5 = 12$

$n = 2$

$S_2 = a_1 + a_2 = 7 \cdot 4 + 5 \cdot 2 = 38$

$a_2 = S_2 - S_1 = 26$

$d = a_2 - a_1 = 14$

II $S_n = \frac{n}{2}(14n + 10)$

$= \frac{n}{2}(24 - 14 + 14n)$

$= \frac{n}{2}(2 \cdot 12 + (n-1) \cdot 14)$

$a_1 = 12$

$d = 14$

$a_2 + a_4 + a_6 + a_8 + a_{10} = 15$

$a_1 + a_2 + a_3 = -3$

) $a_1 + d + a_1 + 3d + a_1 + 5d + a_1 + 7d + a_1 + 9d = 15$ I

$5a_1 + 25d = 15$

$a_1 + 5d = 3$

$a_1 + a_1 + d + a_1 + 2d = -3$

$3a + 3d = -3$

$a_1 + d = -1$

$a_1 + 5d = 3$

$a_1 + d = -1 \quad |(-1)$

$\cancel{a_1} + 5d = 3$

$\cancel{-a_1} - d = 1$

$4d = 4$

$d = 1$

$a_1 = -2$

$\frac{a_2 + a_{10}}{2} = a_6$

$\frac{a_4 + a_8}{2} = a_6$

$\frac{a_1 + a_3}{2} = a_2$

$5a_6 = 15$

$a_6 = 3$

$3a_2 = -3$

$a_2 = -1$

$a_6 = a_2 + 4d$

$4d = 4$

$d = 1$

$a_1 = a_2 - d$

$a_1 = -2$

b) $a_{100} = a_1 + 99d$

$a_{100} = -2 + 99 = 97$

c) $S_{100} = 100 \frac{a_1 + a_{100}}{2}$

$S_{100} = 100 \cdot \frac{-2 + 97}{2}$

$S_{100} = \overset{50}{\cancel{100}} \cdot \frac{95}{\cancel{2}}$

$S_{100} = 4750$

4. $a_3 = 9$

$a_7 - a_2 = 20$

a) $a_1 + 2d = 9$

$\cancel{a_1} + 6d - \cancel{a_1} - d = 20$

$5d = 20$

$d = 4$

$a_1 = 1$

b) $Sn = 91$

$Su = \frac{u}{2}(2a_1 + (u-1)d)$

$91 = \frac{n}{2}(2 + 4(n-1))$

$91 = n + 2u^2 - 2u$

$91 = 2u^2 - n$

$2u^2 - u - 91 = 0$

$n_{1,2} = \frac{1 \pm \sqrt{1 + 728}}{4} \rightarrow u_1 = 7$

$\rightarrow \cancel{u_2 = -6} \quad n \in N$

5. $a_1 = 5$ $\quad a_2 = a_1 + d = 12$

$a_6 = 40$ $\quad a_3 = a_1 + 2d = 19$

I $\quad a_1 + 5d = 40$ $\quad a_4 = a_1 + 3d = 26$

$5 + 5d = 40$ $\quad a_5 = a_1 + 4d = 33$

$5d = 35$

$d = 7$

II $\quad k = 4$ $\quad d = \frac{b-a}{k+1}$ $\quad x_k = a + kd$ $\quad x_3 = 26$

$a = 5$ $\quad d = 7$ $\quad x_1 = 12$ $\quad x_4 = 33$

$b = 40$ $\quad x_2 = 19$

$a_1 + a_2 + a_3 = 27$

$a_1^2 + a_2^2 + a_3^2 = 275$

a) $a_1 = a_2 - d$

$a_3 = a_2 + d$

$a_2 \cancel{-d} + a_2 + a_2 \cancel{+d} = 27$

$3a_2 = 27$

$a_2 = 9$

$a_1 = 9 - d$

$a_3 = 9 + d$

$(9-d)^2 + 9^2 + (9+d)^2 = 275$

$81 \cancel{-18d} + d^2 + 81 + 81 \cancel{+18d} + d^2 = 275$

$2d^2 + 243 = 275$

$2d^2 = 32$

$d^2 = 16$

$d = 4$

) $S = S_{99} - S_9$

$S = \frac{99}{2}(2a_1 + 98d) - \frac{9}{2}(2a_1 + 8d) =$

$= \frac{99}{2}(10 + 98 \cdot 4) - \frac{9}{2}(10 + 8 \cdot 4) =$

$= 19710$

$S_6^u = 6 \cdot \frac{a_1 + a_{11}}{2} = 3 \cdot (a_1 + a_1 + 10d) = 6a_1 + 30d$

$S_6^p = 6 \cdot \frac{a_2 + a_{12}}{2} = 3(a_1 + d + a_1 + 11d) = 6a_1 + 36d$

$S_6^p : S_6^n = 32 : 27$

$27 S_6^p = 32 S_6^u$

$162a_1 + 972d = 192a_1 + 960$

$30a_1 = 12d$

$5a_1 = 2d$

$S_{12} = 354$

$\frac{12}{2}(2a_1 + 11d) = 354$

$12a_1 + 66d = 354 \quad |:6$

$2a_1 + 11d = 59 \quad | \cdot 5$

$\underline{5a_1 - 2d = 0 \quad | \cdot (-2)}$

$\cancel{10a_1} + 55d = 295$

$\underline{\cancel{-10a_1} + 4d = 0}$

$59d = 5 \cdot 59$

$d = 5$

9.2. Geometrijska progresija

- Niz $b_1, b_2, \dots, b_n, \dots$ je geometrijski sa količnikom q ako je
$\frac{b_2}{b_1} = \frac{b_3}{b_2} = \dots \frac{b_n}{b_{n-1}} = \dots = q$

- Uobičajeno je da se za skup konačno mnogo uzastopnih članova geometrijskog niza koristi termin geometrijska progresija.
- Za n-ti član niza važi
$b_n = b_1 q^{n-1} = b_j q^{n-j}, \ j = 2, \dots, n-1$
$b_n^2 = b_{n-1} \cdot b_{n+1} = b_{n-j} \cdot b_{n+j}, \ j = 2, \dots, n-1$
- Zbir prvih n članova geometrijskog niza je
$S_n = b_1 \frac{q^n - 1}{q - 1} = b_1 \frac{1 - q^n}{1 - q}$

- Zbir svih n članova geometrijskog niza je konačan samo za $|q| < 1$ i iznosi $S = \frac{b_1}{1-q}$

- Ako između brojeva a i b treba interpolirati (umetnuti) k brojeva tako da zajedno sa a i b čine geometrijsku progresiju, onda je količnik
$q = \sqrt[k+1]{\frac{b}{a}}$

8. $b_4 = b_2 + 24$
$b_2 + b_3 = 6$

$b_3 = b_2 q$
$b_4 = b_2 q^2$

$b_2 + b_3 = 6$
$b_2 + b_2 q = 6$
$b_2(1+q) = 6$
$b_2 = \frac{6}{1+q}$
$q \neq -1$

$b_4 = b_2 + 24$
$b_2 q^2 - b_2 = 24$
$\frac{6}{1+q} \cdot (q^2 - 1) = 24$
$\frac{6q^2 - 6}{1+q} = 24$

$6q^2 - 6 = 24 + 24q \ | 6$
$q^2 - 4q - 5 = 0$
$q_{1,2} = \frac{4 \pm \sqrt{16+20}}{2} \rightarrow q_1 = 5$
$\rightarrow \cancel{q_2 = -1}$
$b_2 = \frac{6}{6} = 1$
$b_1 = \frac{b_2}{q} = \frac{1}{5}$

9. $b_4 - b_1 = 52$

$b_1 + b_2 + b_3 = 26$

$b_2 = b_1 q$

$b_4 = b_1 q^3$

$b_1 q^3 - b_1 = 52$

$b_1 (q^3 - 1) = 52$

$b_1 (q-1)(q^2+q+1) = 52$

$26(q-1) = 52$

$26q - 26 = 52$

$26q = 78$

$q = 3$

$b_1 + b_1 q + b_1 q^2 = 26$

$b_1 (q^2 + q + 1) = 26$

$b_1 (9 + 3 + 1) = 26$

$b_1 = \frac{26}{13}$

$b_1 = 2$

$S_6 = b_1 \frac{q^6 - 1}{q - 1} = \cancel{2} \cdot \frac{729 - 1}{\cancel{2}}$

$S_6 = 728$

10. $b_1 + b_2 + b_3 + b_4 = 30$

$b_5 + b_6 + b_7 + b_8 = 480$

a) $b_1 (1 + q + q^2 + q^3) = 30$

$b_1 q^4 (1 + q + q^2 + q^3) = 480$

$\frac{30}{\cancel{b_1}} = \frac{480}{\cancel{b_1} q^4}$

$30 q^4 = 480$

$q^4 = 16$

$q = 2$

$b_1 (1 + 2 + 4 + 8) = 30$

$b_1 \cdot 15 = 30$

$b_1 = 2$

) $S_{12} = b_1 \frac{q^{12} - 1}{q - 1} = 2 \cdot \frac{2^{12} - 1}{2 - 1} = 2(2^{12} - 1)$

1. $b_1 + b_2 + b_3 = 13$

$b_1 b_2 b_3 = 27$

a) $b_1 b_2 b_3 = b_1 b_1 q\, b_1 q^2 = b_1^3 q^3 = 27$ $b_1 q = 3$

$b_1 + b_2 + b_3 = b_1(1 + q + q^2) = 13$

$\frac{3}{q}(1 + q + q^2) = 13$

$3 + 3q + 3q^2 = 13q$

$3q^2 - 10q + 3 = 0$

$q_{1,2} = \frac{10 \pm \sqrt{\overset{8}{100 - 36}}}{6} \nearrow q_1 = 3$

$\searrow \cancel{q_2 = \frac{1}{3}}$ - rastući geometrijski niz

$b_1 = \frac{3}{q}$

$b_1 = 1$

b) $S_5 = b_1 \frac{q^5 - 1}{q - 1} = \frac{3^5 - 1}{3 - 1} = \frac{243 - 1}{2} = 121$

12. $b_1 + b_n = \frac{33}{4}$

$b_2 \cdot b_{n-1} = 2$

$b_1 + b_1 q^{n-1} = \frac{33}{4}$

$b_1 q \cdot b_1 q^{n-2} = b_1^2 q^{n-1} = 2$

$q^{n-1} = \frac{2}{b_1^2}$

$q^{n-1} = \frac{\frac{2}{1}}{\frac{1}{16}} = 32$

$b_1 + \cancel{b_1} \cdot \frac{2}{b_1^{\cancel{2}}} = \frac{33}{4}$

$b_1^2 + 2 - \frac{33}{4} b = 0$

$4b_1^2 - 33b + 8 = 0$

$b_{1,2} = \frac{33 \pm \sqrt{\overset{31}{1089 - 128}}}{8} \nearrow \cancel{b_1 = 8}$ - rastući niz

$\searrow \underline{\underline{b_1 = \frac{1}{4}}}$

b) $b_1 \frac{q^n - 1}{q - 1} = \frac{1}{\cancel{4}} \frac{q^n - 1}{q - 1} = \frac{63}{\cancel{4}}$

$q^n - 1 = 63q - 63$

$63q - q^n = 62$

$q(63 - q^{n-1}) = 62$

$q(63 - 32) = 62$

$q \cdot 31 = 62$

$q = 2$

$2^{n-1} = 32$

$2^{n-1} = 2^5$

$n - 1 = 5$

$n = 6$

13. $n = 2k$

$S_{2k} = b_1 \frac{q^{2k} - 1}{q - 1}$

$\frac{b_3}{b_1} = \frac{b_5}{b_3} = \ldots = \frac{b_{2k-1}}{b_{2k-3}} = q^2$

$q_1 = q^2$

$S_k^n = b_1 \cdot \frac{q_1^k - 1}{q_1 - 1} = b_1 \frac{q^{2k} - 1}{q^2 - 1} = b_1 \cdot \frac{q^{2k} - 1}{(q-1)(q+1)}$

$S_{2k} = 5 S_k^n \Leftrightarrow \cancel{b_1} \frac{\cancel{q^{2k} - 1}}{\cancel{q - 1}} = 5 \cancel{b_1} \cdot \frac{\cancel{q^{2k} - 1}}{\cancel{(q-1)}(q+1)}$.

$= \frac{5}{q+1} \quad q+1=5 \quad q=4$

b)

$a_1 = b_1 + 3$
$a_2 = b_2 + 1$
$a_3 = b_3 - 5$

a) $\frac{a_1+a_3}{2} = a_2$
$a_1 + a_3 = 2a_2$
$b_1 + 3 + b_3 - 5 = 2b_2 + 2$
$b_1 + b_3 = 2b_2 + 4$

$b_1 + b_2 + b_3 = 28$
$2b_2 + 4 + b_2 = 28$
$3b_2 = 24$
$b_2 = 8$

$b_1 = \frac{b_2}{q} = \frac{8}{q}$
$b_3 = b_2 q = 8q$
$b_1 + b_3 = \frac{8}{q} + 8q =$
$= 16 + 4 = 20$
$8 + 8q^2 - 20q = 0 \quad |:4$
$2q^2 - 5q + 2 = 0$
$q_{1,2} = \frac{5 \pm \sqrt{25-16}}{4} \xrightarrow{} q_1 = 2$
$\searrow q_2 = \frac{1}{2}$

16

$_1 = \frac{8}{q_1} = 4$
$_3 = 8q_1 = 16$
$_2 = 8$

V

$b_1 = \frac{8}{q_2} = 16$
$b_3 = 8q_2 = 4$
$b_2 = 8$

1)

2)

$b_1 = 16,\ b_2 = 8,\ b_3 = 4 \rightarrow a_1 = 19,\ a_2 = 9,\ a_3 = -1$
$b_1 = 4,\ b_2 = 8,\ b_3 = 16 \rightarrow a_1 = 7,\ a_2 = 9,\ a_3 = 11$

17.

$b_1 = a_1 + 8$
$b_2 = a_2$
$b_3 = a_3$
$b_1 + b_2 + b_3 = 26$
$a_1 + a_3 = 2a_2$

a) $b_1 + b_2 + b_3 = a_1 + 8 + a_2 + a_3 =$
$= 2a_2 + a_2 + 8$
$3a_2 + 8 = 26$
$3a_2 = 18$
$a_2 = 6$

$a_1 = a_2 - d = 6 - d$
$b_1 = 6 - d + 8 = 14 - d$
$a_3 = a_2 + d = 6 + d$
$\frac{b_2}{b_1} = \frac{b_3}{b_2} = \frac{6}{14-d} = \frac{6+d}{6}$
$84 + 14d - 6d - d^2 = 36$
$d^2 - 8d - 48 = 0$
$d_{1,2} = \frac{8 \pm \sqrt{64+192}}{2} \nearrow d_1 = 12$
$\searrow d_2 = -4$

$_1 = 12 \rightarrow a_1 = -6,\ a_2 = 6,\ a_3 = 18$
$= -4 \rightarrow a_1 = 10,\ a_2 = 6,\ a_3 = 2$

-

b) 1) $b_1 = 2$, $b_2 = 6$, $b_3 = 18$

2) $b_1 = 18$, $b_2 = 6$, $b_3 = 2$

16. $a_2 = a_1 q$

$a_3 = a_1 q^2$

$a_2 + a_4 = 2a_3$

$a_1 q + a_4 = 2a_1 q^2$

$a_1 + a_4 = 14$

$a_4 = 14 - a_1$

$a_1 q + 14 - a_1 = 2a_1 q^2$

$a_1(2q^2 - q + 1) = 14$

$a_2 + a_3 = 12$

$a_1 q + a_1 q^2 = 12$

$a_1(q + q^2) = 12$

$a_1 = \frac{12}{q + q^2}$

$\frac{12}{q+q^2}(2q^2 - q + 1) = 14$ $q \neq 0, q \neq -1$

$2q^2 - q + 1 = \frac{14q + 14q^2}{12} \quad \Big| \cdot 12$

$24q^2 - 12q + 12 = 14q + 14q^2$

$10q^2 - 26q + 12 = 0 \quad \Big| : 2$

$5q^2 - 13q + 6 = 0$

$q_{1,2} = \frac{13 \pm \overset{7}{\sqrt{169 - 120}}}{10} \begin{matrix} \nearrow q_1 = 2 \\ \searrow q_2 = \frac{3}{5} \end{matrix}$

2

$\frac{1}{2}$

1) $q = 2 \rightarrow a_1 = \frac{12}{4+2} = 2$, $a_2 = 4$, $a_3 = 8$, $a_4 = 12$

2) $q = \frac{3}{5} \rightarrow a_1 = \frac{12}{\frac{3}{5} + \frac{9}{25}} = \frac{25}{2}$, $a_2 = \frac{15}{2}$, $a_3 = \frac{9}{2}$, $a_4 = \frac{3}{2}$

17. $S = \frac{b_1}{1-q} = 32$, $|q| < 1$

$b_1 = 32(1-q)$

$b_1 - b_2 = 8$

$b_1 - b_1 q = 8$

$b_1(1-q) = 8$

$b_1 = \frac{8}{1-q}$

$32(1-q) = \frac{8}{1-q}$

$32(1-q)^2 = 8$

$(1-q)^2 = \frac{1}{4}$

$1 - q = \pm\frac{1}{2}$

$\cancel{q = \frac{3}{2}}$, $q = \frac{1}{2}$

$b_1 = \frac{8}{1-\frac{1}{2}} = 16$

$b_1 q^{n-1} = 1$

$16 q^{n-1} = 1$

$16\left(\frac{1}{2}\right)^{n-1} = 1$

$2^{n-1} = 16$

$n - 1 = 4$

$n = 5$

1.3. Zadaci za vježbu

. $a_2+a_5=8$

$a_3+a_7=14$

$a_1+d+a_1+4d=8$

$2a_1+5d=8 \quad |\cdot(-1)$

$a_1+2d+a_1+6d=14$

$2a_1+8d=14$

$-\cancel{2a_1}-5d=-8$

$\cancel{2a_1}+8d=14$

$3d=6$

$d=2$

$a_1=-1$

2. $a_1=8$

$a_9=88$

$a_1+8d=88$

$8+8d=88$

$8d=80$

$d=10$

$a_2=18$

$a_3=28$

$a_4=38$

$a_5=48$

$a_6=58$

$a_7=68$

$a_8=78$

3. $a_3=9$

$a_7-a_2=20$

$S_n=91$

$a_1+2d=9$

$\cancel{a_1}+6d-\cancel{a_1}-d=20$

$5d=20$

$d=4$

$a_1=1$

$S_n=\frac{n}{2}(2a_1+(n-1)\cdot d)$

$91=\frac{n}{2}(2+(n-1)\cdot 4)$

$182=n(2+4n-4)$

$2n+4n^2-4n=182$

$4n^2-2n-182=0 \quad |:2$

$2n^2-n-91=0$ $\quad 27$

$n_{1,2}=\frac{1\pm\sqrt{1+728}}{4} \rightarrow n_1=7$

$\rightarrow \cancel{n_2=-\frac{26}{4}}$

$n=7$

4. $a_1+a_2+a_3=18$

$a_1 a_2 a_3 = 162$

$a_2 = \frac{a_1+a_3}{2}$

$2a_1+a_1+a_3+2a_3=36$

$3a_1+3a_3=36$

$3(a_1+a_3)=36$

$a_1+a_3=12$

$a_2=\frac{12}{2}=6$

$a_1=12-a_3$

$a_1 \cdot a_3 = 27$

$(12-a_3)a_3=27$

$12a_3-a_3^2-27=0$

$a_{3_{1,2}} = \frac{-12 \pm \sqrt{144-108}}{-2} \rightarrow a_3=3$

$\rightarrow a_3=9$

$a_3=3 \rightarrow a_2=9$

$a_3=9 \rightarrow a_2=3$

$d=3$

$a_1=3,\ a_2=6,\ a_3=9$

5. $\frac{b_4}{b_6}=\frac{1}{4}$

$b_2+b_5=216$

$b_1 q+b_1 q^4=216$

$b_1(q+q^4)=216$

$\frac{b_6}{b_4}=4$

$\frac{\cancel{b_1}q^5}{\cancel{b_1}q^3}=4$

$q^2=4$

$q=\pm 2$

$b_1(2+16)=216 \ \wedge \ b_1(-2+16)=216$

$b_1 \cdot 18=216 \qquad b_1 \cdot 14=216$

$b_1=12 \qquad b_1=\frac{216}{14}=\frac{108}{7}$

6. $b_1+b_3=20$

$b_1+b_2+b_3=26$

$20+b_2=26$

$b_2=6$

$b_1 q=6$

$b_1=\frac{6}{q}$

$\frac{6}{q}+b_1 \cdot q^2=20$

$\frac{6}{q}+\frac{6}{\cancel{q}} \cdot \cancel{q^2}=20$

$\frac{6}{q}+6q=20 \ |\cdot q$

$6+6q^2=20q \ |:2$

$3q^2-10q+3=0$

$q_{1,2}=\frac{10 \pm \sqrt{100-36}}{6} \rightarrow q_1=3$

$\rightarrow q_2=\frac{1}{3}$

1) $q=3 \rightarrow b_1=2,\ b_2=6,\ b_3=18$

2) $q=\frac{1}{3} \rightarrow b_1=18,\ b_2=6,\ b_3=2$

7. $d \neq 0$

$a_2 = b_1$

$a_1 = b_2$

$a_3 = b_3$

$a_1 + a = b_1$

$a_1 = b_1 q$

$a_1 + 2d = b_1 q^2$

$d = b_1 - a_1$

$2d = b_1 q^2 - a_1$

$d = \frac{b_1 q^2 - a_1}{2}$

$b_1 - a_1 = \frac{b_1 q^2 - a_1}{2}$

$2b_1 - 2b_1 q = b_1 q^2 - b_1 q$

$2b_1 - b_1 q^2 = b_1 q$

$2b_1 = b_1 q^2 + b_1 q$

$2\cancel{b_1} = \cancel{b_1}(q^2 + q)$

$q^2 + q - 2 = 0$

$q_{1,2} = \frac{-1 \pm \overset{3}{\sqrt{1+8}}}{2}$ $\rightarrow \cancel{q_1 = 1}$ $\rightarrow q_2 = -2$

) $q = 1 \rightarrow d = b_1 - b_1 q$

$d = b_1(1 - q)$

$\cancel{d = b_1 \cdot 0} \rightarrow d \neq 0$

) $q = -2$ ✓

$q = -2$

. $a_1 + a_2 + a_3 = 15$

$a_5 + a_7 + a_9 = 60$

$a_2 = \frac{a_1 + a_3}{2}$

$2a_1 + a_1 + a_3 + 2a_3 = 15$

$3(a_1 + a_3) = 30$

$a_1 + a_3 = 10$

$a_2 = 5$

$a_7 = \frac{a_5 + a_9}{2}$

$2a_5 + a_5 + a_9 + 2a_9 = 120$

$3(a_5 + a_9) = 120$

$a_5 + a_9 = 40$

$a_7 = 20$

$a_1 + d = 5 \quad |(-1)$

$\underline{a_1 + 6d = 20}$

$a_1 - d = -5$

$\underline{a_1 + 6d = 20}$

$5d = 15$

$d = 3$

$a_{10} = a_1 + 9d$

$a_{10} = 2 + 27$

$a_{10} = 29$

. I $a_1 = 38$ II $a_1 = 11$

$a_4 = 13$ $\quad a_7 = 35$

$a_4 = a_4$

II $a_7 = a_1 + 6d$

$35 = 11 + 6d$

$6d = 24$

$d = 4$

$a_4 = a_1 + 3d$

$a_4 = 11 + 12$

$a_4 = 23$

I $a_4 = a_1 + 3d$

$3d = 23 - 38$

$3d = -15$

$d = -5$

$a_n = a_1 + (n-1)d$

$13 = 38 + (n-1)\cdot(-5)$

$(n-1)(-5) = -25$

$n-1 = \frac{-25}{-5}$

$n-1 = 5$

$n = 6$

10. $S_{10}^{u} = 220$ $\qquad S_{10}^{u} = 10 \cdot \frac{a_1 + a_{19}}{2} = 5(a_1 + a_1 + 18d) = 10a_1 + 90d$

$S_{10}^{p} = 250$ $\qquad S_{10}^{p} = 10 \cdot \frac{a_2 + a_{20}}{2} = 5(a_1 + d + a_1 + 19d) = 10a + 1000d$

$10a_1 + 90d = 220 \quad |(-1)$

$\underline{10a_1 + 100d = 250}$

$-\cancel{10a_1} - 90d = -220$

$\underline{\cancel{10a_1} + 100d = 250}$

$10d = 30$

$d = 3$

11. $a_1 = \frac{1}{\log_3 2} = \log_2 3$ $\qquad a_2 = \frac{a_1 + a_3}{2}$

$a_2 = \frac{1}{\log_6 2} = \log_2 6$ $\qquad \log_2 6 = \frac{\log_2 3 + \log_2 12}{2}$

$a_3 = \frac{1}{\log_{12} 2} = \log_2 12$ $\qquad 2\log_2 6 = \log_2(3\cdot 12) = \log_2 36$

$\log_2 6^2 = \log_2 36$

$\log_2 36 = \log_2 36$

12. $a_1 = \ln 2$

$a_2 = \ln(2^x - 1)$

$a_3 = \ln(2^x + 3)$

$a_2 = \frac{a_1 + a_3}{2}$

$\ln(2^x - 1) = \frac{\ln 2 + \ln(2^x + 3)}{2}$

$2\ln(2^x - 1) = \ln(2^{x+1} + 6)$

$(2^x - 1)^2 = (2^{x+1} + 6)$

$2^{2x} - 2^{x+1} + 1 = 2^{x+1} + 6$

$2^{2x} - 2^{x+2} = 5$

$2^{2x} - 2^x \cdot 4 = 5 \qquad 2^x = t$

$t^2 - 4t - 5 = 0$

$t_{1,2} = \frac{4 \pm \overset{6}{\sqrt{16+20}}}{2} \rightarrow t_1 = 5$

$\rightarrow \cancel{t_2 = -1} \qquad t > 0$

$2^x = 5$

$x = \log_2 5$

13. $b_1 = 7$

$b_{\frac{n+1}{2}} = 56$

$S_n = 889$

$b_{\frac{n+1}{2}} = b_1 \cdot q^{\frac{n-1}{2}}$

$q^{\frac{n-1}{2}} = 8$

$S_n = b_1 \frac{q^n - 1}{q - 1}$

$\frac{q^n - 1}{q - 1} = 127$

$\frac{(q^{\frac{n-1}{2}})^2 \cdot q - 1}{q - 1} = 127$

$\frac{64q - 1}{q - 1} = 127$

$64q - 1 = 127q - 127$

$63q = 126$

$q = 2$

$q^{\frac{n-1}{2}} = q^3$

$\frac{n-1}{2} = 3$

$n - 1 = 6$

$n = 7$

14. $q = 2$

$S_{k_1} = 15$

$S_{k_2} = 240$

$S_{k_1} = b_1 \frac{q^k - 1}{q - 1}$

$S_{k_1} = b_1 \cdot \frac{q^k - 1}{1}$

$S_{k_1} = b_1(q^k - 1)$

$b_1(q^k - 1) = 15$

$S_{2k} = S_{k_1} + S_{k_2} = 255$

$S_{2k} = b_1 \frac{q^{2k} - 1}{q - 1}$

$b_1(q^{2k} - 1) = 255$

$S_{2k} = 17 S_{k_1}$

$\cancel{b_1}(q^{2k} - 1) = 17\cancel{b_1}(q^k - 1)$

$q^{2k}-1=17q^k-17$

$q^{2k}-17q^k+16=0$ $q^k=t$

$t^2-17t+16=0$

$t_{1,2}=\frac{17\pm\sqrt{\overset{15}{289-64}}}{2}$ $\nearrow t_1=16$, $\searrow t_2=1$

1) $2^k=16$ 2) $2^k=1$

$2^k=2^4$ $2^k=2^0$

$\underline{\underline{k=4}}$ ~~$k=0$~~

$2k=\underline{\underline{8}}$

$b_1=\frac{S_k}{(q^k-1)}$

$b_1=\frac{15}{15}$

$b_1=1$

$b_8=b_1\cdot q^7$

$b_8=1\cdot 2^7$

$b_8=128$

15. $b_1+b_2+b_3=\frac{3}{2}$

$\frac{1}{b_1}+\frac{1}{b_2}+\frac{1}{b_3}=\frac{3}{2}$

$\frac{b_2b_3+b_1b_3+b_1b_2}{b_1b_2b_3}=\frac{3}{2}$

$\frac{b_1q\,b_1q^2+b_1\,b_1q^2+b_1\,b_1q}{b_1\cdot b_1\cdot q\cdot b_1\cdot q^2}=\frac{3}{2}$

$\frac{b_1^2q^3+b_1^2q^2+b_1^2q}{b_1^3q^3}=\frac{3}{2}$

$\frac{\cancel{b_1^2q}(q^2+q-1)}{b_1^3q^{\cancel{3}2}}=\frac{3}{2}$

$\frac{q^2+q+1}{b_1q^2}=\frac{3}{2}$

$S_3=b_1\cdot\frac{q^3-1}{q-1}=\frac{\cancel{(q-1)}(q^2+q+1)}{\cancel{q-1}}$

$b_1(q^2+q+1)=\frac{3}{2}$

$b_1(\cancel{q^2+q+1})=\frac{\cancel{q^2+q+1}}{b_1q^2}$

$b_1=\frac{1}{b_1q^2}$

$b_1^2q^2=1$

$(b_1q)^2=1$

$b_2^2=1$

$b_2=\pm1$

$b_3=b_1\cdot q^2$ $q^2=\frac{1}{b_1^2}$

$b_3=\cancel{b_1}\frac{1}{b_1^{\cancel{2}}}$

$b_3=\frac{1}{b_1}$

I Ako je $b_2=1$

$b_1+1+\frac{1}{b_1}=\frac{3}{2}$

$\frac{b_1^2+b_1+1}{b_1}=\frac{3}{2}\quad\Big|\cdot 2b_1$

$2b_1^2+2b_1+2=3b_1$

$2b_1^2-b_1+2=0$

$b_{1,2}=\frac{1\pm\sqrt{1-16}}{4}\notin R$

Znači da b_2 nije 1, odnosno $b_2=-1$.

$b_2 = -1$

$b_1 - 1 + \frac{1}{b_1} = \frac{3}{2}$

$\frac{b_1^2 - b_1 + 1}{b_1} = \frac{3}{2} \quad / \cdot 2b_1$

$2b_1^2 - 2b_1 + 2 = 3b_1$

$2b_1^2 - 5b_1 + 2 = 0$

$b_{1,2} = \frac{5 \pm \overset{3}{\sqrt{25-16}}}{4} \begin{cases} \rightarrow b_1 = 2 \\ \rightarrow \cancel{b_1 = \frac{1}{2}} \end{cases}$

Pošto je niz opadajući, a $b_3 = \frac{1}{b_1}$, sledi da b_1 mora biti 2.

$b_1 = 2,\ b_2 = -1,\ b_3 = \frac{1}{2}$

6. $S_3 = 91$

$(b_1 + b_3) \cdot \frac{30}{61} = b_2$

$(b_1 + b_1 q^2) \cdot \frac{30}{61} = b_1 \cdot q$

$\cancel{b_1}(1 + q^2) \cdot \frac{30}{61} = \cancel{b_1} \cdot q$

$(1 + q^2) \cdot \frac{30}{61} = q$

$30 + 30q^2 = 61q$

$30q^2 - 61q + 30 = 0$

$q_{1,2} = \frac{61 \pm \overset{11}{\sqrt{3721 - 3600}}}{60} \begin{cases} \rightarrow \cancel{q_1 = \frac{6}{5}} \rightarrow \text{niz je opadajući.} \\ \rightarrow \underline{\underline{q_2 = \frac{5}{6}}} \end{cases}$

$91 = b_1 \cdot \frac{q^3 - 1}{q - 1} = b_1 \cdot \frac{\cancel{(q-1)}(q^2 + q + 1)}{\cancel{q-1}}$

$91 = b_1 \cdot (q^2 + q + 1)$

$91 = b_1 \cdot \left(\frac{25}{36} + \frac{5}{6} + 1\right)$

$91 = b_1 \cdot \left(\frac{25 + 30 + 36}{36}\right)$

$\cancel{91} = b_1 \cdot \left(\frac{\cancel{91}}{36}\right)$

$b_1 = 36$

$b_2 = 30$

$b_3 = 25$

7. $b_1 = 1$

$b_5 = 256$

$q = \sqrt[4]{\frac{b_5}{b_1}}$

$q = \sqrt[4]{256}$

$q = 4$

$b_2 = b_1 \cdot q = 4$

$b_3 = b_1 \cdot q^2 = 16$

$b_4 = b_1 \cdot q^3 = 64$

18. $C_4 = 500$

$C_4 = 125\% \cdot C_3$

$C_3 = \frac{50000}{125}$

$C_3 = 400$

$C_3 = 125\% \cdot C_2$

$C_2 = \frac{40000}{125}$

$C_2 = 320$

$C_2 = 125\% \cdot C_1$

$C_1 = \frac{32000}{125}$

$C_1 = 256$

I. $x + x + \frac{25}{100}x + \left(x + \frac{25}{100}x\right) + \frac{25}{100}\left(x + \frac{25}{100}x\right) \ldots = 500$

$p = 25\%$ $\quad x + (p+1)x + (p+1)^2 x + (p+1)^3 x = 500$

$p+1 = 2$

$x = a$

$n = 4$

$S_n = 500$

19. $a_1 = \frac{1}{\log_3 5} = \log_5 3$

$a_2 = \frac{1}{\log_6 5} = \log_5 6$

$a_3 = \frac{1}{\log_x 5} = \log_5 x$

$2a_2 = a_1 + a_3$

$2\log_5 6 = \log_5 3 + \log_5 x$

$\log_5 36 = \log_5 (3x)$

$3x = 36$

$x = 12$

20. $a_2 - 2 = b_2$

$a_1 = b_1$

$a_3 = b_3$

$b_1 + b_2 + b_3 = 28$

$S_3 = b_1 \cdot \frac{q^3 - 1}{q - 1} = b_1 \cdot \frac{\cancel{(q-1)}(q^2+q+1)}{\cancel{q-1}}$

$28 = b_1 \cdot (q^2 + q + 1)$

$2a_2 = a_1 + a_3$

$2(b_2 + 2) = b_1 + b_3$

$2b_2 + 4 = b_1 + b_3$

$2b_1q + 4 = b_1 + b_1q^2$

$b_1 + b_1q^2 - 2b_1q - 4 = 0$

$b_1 + b_1q^2 + b_1q - 3b_1q - 4 = 0$

$b_1 \cdot (q^2 + q + 1) - 3b_1q - 4 = 0$

$28 - 3b_1q - 4 = 0$

$3b_2 = 24$

$b_2 = 8$

$a_2 = 10$

$2a_2 = a_1 + a_3$

$20 = a_1 + a_3$

$a_3 = 20 - a_1$

${}^2 = b_1 \cdot b_3$

$4 = a_1 \cdot a_3$

$64 = a_1(20 - a_1)$

$64 = 20a_1 - a_1^2$

$a_1^2 - 20a_1 + 64 = 0$

$$a_{1,3} = \frac{20 \pm \overset{12}{\sqrt{400 - 256}}}{2} \xrightarrow{} a_1 = 4$$

$$\searrow a_3 = 16$$

$a_2 = 10$

1\. $S_3 = 63$

$a_1 - 7 = b_1$

$a_2 - 9 = b_2$

$a_3 - 5 = b_3$

$S_3 = 3 \cdot \frac{a_1 + a_3}{2}$

$S_3 = 3a_2$

$a_2 = \frac{63}{3}$

$a_2 = 21$

$a_1 + a_3 = 42$

$b_1 \cdot b_3 = b_2^2$

$(a_1 - 7)(a_3 - 5) = (a_2 - 9)^2$

$(a_1 - 7)(42 - a_1 - 5) = 144$

$(a_1 - 7)(37 - a_1) = 144$

$37a_1 - a_1^2 - 259 + 7a_1 = 144$

$a_1^2 - 44a_1 + 403 = 0$

$$a_{1,2} = \frac{44 \pm \overset{18}{\sqrt{1936 - 1600}}}{2}$$

$a_1 = 31$

~~$a_1 = 13$~~

opadajuća progresije

$a_1 = 31$

$a_2 = 21$

$a_3 = 11$

2\. $S_u = \frac{b_1}{1 - q} = -8$

$b_4 - b_2 = 6(q - 1)^2$

$b_1 = 8(q - 1)$

$b_1 \cdot q^3 - b_1 q = 6 \cdot (q - 1)^2$

$b_1 \, q(q^2 - 1) = 6 \cdot (q - 1)^2$

$b_1 \cdot q \cdot \cancel{(q - 1)} \cdot (q + 1) = 6 \cdot (q - 1)^{\cancel{2}}$

$8\cancel{(q - 1)} \cdot q(q + 1) = 6 \cdot \cancel{(q - 1)}$

$8q^2 + 8q - 6 = 0 \quad | : 2$

$4q^2 + 4q - 3 = 0$

$$q_{1,2} = \frac{-4 \pm \overset{8}{\sqrt{16 + 48}}}{8} \xrightarrow{} \cancel{q_1 = -\frac{12}{8}} \quad |q| < 1$$

$$\searrow q_2 = \frac{1}{2}$$

$b_1 = 8\left(\frac{1}{2} - 1\right)$

$b_1 = \overset{4}{\cancel{8}} \cdot \left(-\frac{1}{\cancel{2}}\right)$

$b_1 = -4$

$b_2 = -2$

23. $a_1 + a_2 = 20$

$a_3 + a_4 = 6$

$a_3 = a_2 q$

$a_4 = a_2 q^2$

$2a_2 = a_1 + a_3$

$2a_2 = a_1 + a_2 q$

$2a_2 = 20 - a_2 + a_2 q$

$2a_2 = 20 - a_2(1-q)$

$20 = 2a_2 + a_2(1-q)$

$20 = a_2(2+1-q)$

$20 = a_2(3-q)$

$a_2 = \frac{20}{3-q}$

$a_2 q + a_2 q^2 = 6$

$a_2(q+q^2) = 6$

$\frac{20}{3-q}(q+q^2) = 6$

$20q^2 + 20q = 18 - 6q$

$20q^2 + 26q - 18 = 0 \quad |:2$

$10q^2 + 13q - 9 = 0$

$$q_{1,2} = \frac{-13 \pm \overset{23}{\sqrt{169+360}}}{20} \begin{cases} \nearrow q_1 = \frac{1}{2} \\ \searrow \cancel{q_2 = -\frac{9}{5}} \end{cases}$$

$a_2 = \frac{20}{3-\frac{1}{2}} = 8$

$a_1 = 12$

$a_3 = 4$

$a_4 = 2$

24. $\frac{b_1}{1-q} = -4$

$b_4 - b_1 = 7(q-1)^2$

$b_1 q^3 - b_1 = 7(q-1)^2$

$b_1(q^3-1) = 7(q-1)^2$

$b_1 \cdot \cancel{(q-1)}(q^2+q+1) = 7(q-1)^{\cancel{2}}$

$4\cancel{(q-1)}(q^2+q+1) = 7\cancel{(q-1)}$

$4q^2 + 4q + 4 = 7$

$4q^2 + 4q - 3 = 0$

$$q_{1,2} = \frac{-4 \pm \overset{8}{\sqrt{16+48}}}{8} \begin{cases} \nearrow q_1 = \frac{1}{2} \\ \searrow \cancel{q_2 = \frac{-12}{8}} \quad |q| < 1 \end{cases}$$

$b_1 = -4(1-q)$

$b_1 = 4(q-1)$

$b_1 = 4\left(\frac{1}{2} - 1\right)$

$b_1 = -2$

$b_{2002} = b_1 \cdot q^{2001}$

$b_{2002} = -2\left(\frac{1}{2}\right)^{2001}$

$b_{2002} = -2 \cdot 2^{-2001}$

$b_{2002} = -2^{-2000}$

26. $a_1 = \frac{1}{\log_3 5} = \log_5 3$

$a_2 = \frac{1}{\log_6 5} = \log_5 6$

$a_3 = \frac{1}{\log_x 5} = \log_5 x$

$2a_2 = a_1 + a_3$

$2\log_5 6 = \log_5 3 + \log_5 x$

$\log_5 6^2 = \log_5 (3x)$

$\log_5 36 = \log_5 (3x)$

$3x = 36$

$x = 12$

10. Matematička indukcija i kombinatorika

10.1. Matematička indukcija

- Ako za neko tvrđenje $T(n)$, $n \in N$ važi:

1) $T(1)$ je tačno
2) $T(n) \Rightarrow T(n+1)$ je tačno za svako $n = 1, 2, \dots$

tada je tvrđenje $T(n)$ tačno za svako $n \in N$

- Uopštenje 1:

Ako za neko tvrđenje $T(n)$, $n \geq k$, $n, k \in N$, važi:

1) $T(k)$ je tačno,
2) $T(n) \Rightarrow T(n+1)$ je tačno za svako $n = k, k+1, \dots$,

tada je tvrđenje $T(n)$ tačno za svako $n \geq k$.

- Uopštenje 2:

Ako za neko tvrđenje $T(n)$, $n \in N$, važi:

1) $T(1), T(2), \dots, T(k)$ je tačno,
2) $T(n-k+1), \dots, T(n-1), T(n) \Rightarrow T(n+1)$ je tačno za svako $n = k, k+1, \dots$,

tada je tvrđenje $T(n)$ tačno za svako $n \in N$.