

8. Stereometrija

8.1. Prizma, valjak i sfera

- Zapremina prizme baze B i visine $V = B \cdot h$
- Površina prizme baze B i omotača M $P = 2B + M$
- Prizma je prava ako je visina normalna na ravni osnove (baze)
- Prizma je pravilna ako je njena baza pravilni mnogougao
- * Kocka $P = 2(ab+ac+bc)$, $V = a \cdot b \cdot c$
- * Kocka $P = 6a^2$, $V = a^3$

- Zapremina valjka visine H i poluprečnika osnove r

$$V = B \cdot H = r^2 \pi H$$

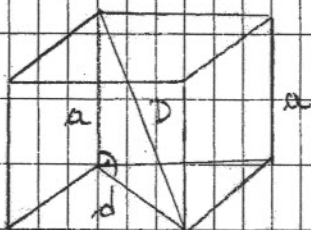
- Površina pravog valjka visine H i poluprečnika osnove r

$$P = 2B + M = 2r^2 \pi + 2r \pi H = 2r \pi (r + H)$$

- Površina sfere (lopte) poluprečnika R $P = 4R^2 \pi$

- Zapremina sfere (lopte) poluprečnika R $V = \frac{4R^3 \pi}{3}$

1.



$$D^2 = a^2 + d^2$$

$$D = 2R$$

$$P = 6 \frac{4R^2}{3}$$

$$D^2 = a^2 + (a\sqrt{2})^2$$

$$4R^2 = 3a^2$$

$$P = 8R^2$$

$$D^2 = a^2 + 2a^2$$

$$a^2 = \frac{4R^2}{3}$$

$$D^2 = 3a^2$$

$$P = 6a^2$$

2. $r = 2$

a) $r = \frac{2}{3} h$

$$h = \frac{a\sqrt{3}}{2}$$

$$B = \frac{a^2 \sqrt{3}}{4}$$

b) $a_1 = 2$

$$H = \frac{V}{B} = \frac{8}{3\sqrt{3}}$$

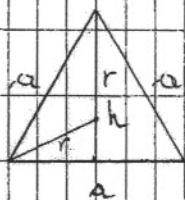
$$h = \frac{\frac{2}{3} h}{\frac{2}{3}}$$

$$a = \frac{2h}{\sqrt{3}}$$

$$B = \frac{4 \cdot 3\sqrt{3}}{4}$$

$$V_1 = a_1^3$$

$$H = \frac{8\sqrt{3}}{9}$$



$$h = \frac{3r}{2}$$

$$a = \frac{6\sqrt{3}}{3}$$

$$B = 3\sqrt{3}$$

$$V = 8$$

$$M = 3aH$$

$$h = 3$$

$$a = 2\sqrt{3}$$

$$V_1 = V$$

$$V = B \cdot H$$

$$M = 8 \cdot 2\sqrt{3} = \frac{8\sqrt{3}}{3}$$

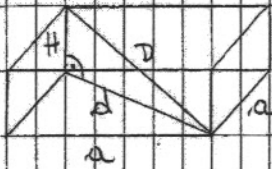
$$M = 16$$

$$P = 2B + M$$

$$P = 6\sqrt{3} + 16$$

$$3. \quad H = 2 \quad a) \quad B = a^2 \quad M = 4aH \quad P = 2B + M \quad V = BH$$

$$a = 4 \quad B = 16 \quad M = 32 \quad P = 64 \quad V = 32$$



$$D^2 = d^2 + H^2$$

$$D = 2R$$

$$V = \frac{4}{3} R^3 \pi$$

$$D^2 = (a\sqrt{2})^2 + H^2$$

$$R = 3$$

$$V = \frac{4}{3} \cdot 27 \pi$$

$$D^2 = 32 + 4$$

$$P = 4R^2 \pi$$

$$V = 36\pi$$

$$D^2 = 36$$

$$P = 36\pi$$

$$D = 6$$

$$4. \quad a = 3 \quad a) \quad D^2 = H^2 + a^2 \quad B = \frac{3a^2\sqrt{3}}{2} \quad P = 2B + M \quad V = \frac{243}{2}$$

$$d = 6$$

$$H^2 = 36 - 9$$

$$B = \frac{27\sqrt{3}}{2}$$

$$P = 24\sqrt{3} + 54\sqrt{3}$$

$$H^2 = 27$$

$$B = \frac{27\sqrt{3}}{2}$$

$$P = 81\sqrt{3}$$

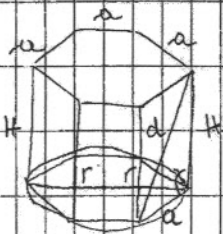
$$H = 3\sqrt{3}$$

$$M = 6aH$$

$$V = BH$$

$$M = 54\sqrt{3}$$

$$V = \frac{27\sqrt{3}}{2} \cdot 3\sqrt{3}$$



$$b) \quad r = a$$

$$M = 2r\pi H$$

$$P = 2B + M$$

$$B = r^2\pi$$

$$M = 18\sqrt{3}\pi$$

$$P = 18\pi + 18\sqrt{3}\pi$$

$$V = BH$$

$$B = 9\pi$$

$$P = 18\pi(1 + \sqrt{3})$$

$$V = 27\sqrt{3}\pi$$

$$5. \quad a = 4 \quad a^2 = \left(\frac{d_1}{2}\right)^2 + \left(\frac{d_2}{2}\right)^2 \quad P = \frac{d_1 d_2}{2} \quad V = BH$$

$$d_2 = 6$$

$$\left(\frac{d_1}{2}\right)^2 = 16 - 9$$

$$P = \frac{12\sqrt{7}}{2}$$

$$V = \frac{a^2\sqrt{3}}{4} \cdot 4$$

$$\left(\frac{d_1}{2}\right)^2 = 7$$

$$P = 6\sqrt{7}$$

$$V = \frac{16\sqrt{3}}{4} \cdot \frac{3\sqrt{7}}{2}$$

$$\frac{d_1}{2} = \sqrt{7}$$

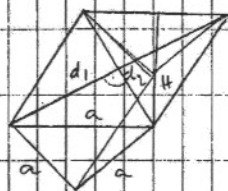
$$P = aH$$

$$V = 6\sqrt{21}$$

$$d_1 = 2\sqrt{7}$$

$$H = \frac{6\sqrt{7}}{4}$$

$$H = \frac{3\sqrt{7}}{2}$$

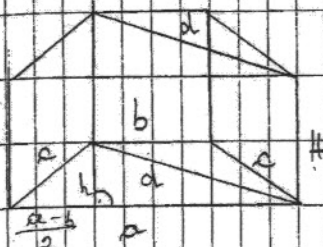


6. $a = 21$
 $b = 11$
 $c = 13$
 $P_d = 180$

a) $c^2 = h^2 + \left(\frac{a-b}{2}\right)^2$
 $h^2 = 169 - 25$
 $h^2 = 144$
 $h = 12$

$d^2 = h^2 + \left(a - \frac{a+b}{2}\right)^2$
 $d^2 = 144 + \left(\frac{a+b}{2}\right)^2$
 $d^2 = 144 + 256$
 $d^2 = 400$
 $d = 20$

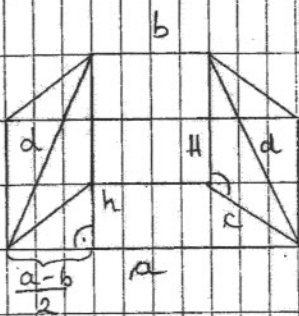
$P_d = d \cdot H$
 $H = \frac{180}{20}$
 $H = 9$
 $B = \frac{a+b}{2} \cdot h$



$P = 2B + M$
 $P = 384 + 522$
 $P = 906$
 $V = B \cdot H$
 $V = 192 \cdot 9$
 $V = 1728$

$B = 192$
 $U = (a+b+2c) \cdot H$
 $U = 58 \cdot 9$
 $U = 522$

b)



$d^2 = h^2 + c^2$
 $d^2 = 250$
 $d = 5\sqrt{10}$
 $d^2 = h^2 + \left(\frac{a-b}{2}\right)^2$
 $h^2 = 250 - 25$
 $h^2 = 225$
 $h = 15$

$P = \frac{a+b}{2} \cdot h$
 $P = 16 \cdot 15$
 $P = 240$

H. $H = 2r$
 $U = 80$

$M = 2r\pi H$
 $M = 4r^2\pi$
 $r^2 = \frac{M}{4\pi}$

$r^2 = \frac{80}{4\pi}$
 $r^2 = \frac{20}{\pi}$
 $r = \sqrt{\frac{20}{\pi}}$
 $r = \frac{2\sqrt{5}}{\sqrt{\pi}}$

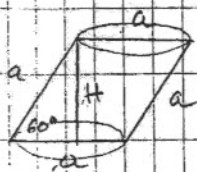
$B = \pi r^2$
 $B = \frac{20}{\pi} \cdot \pi$
 $B = 20$

$P = 2B + M$
 $P = 40 + 80$
 $P = 120$
 $V = B \cdot H$
 $V = 20 \cdot \frac{4\sqrt{5}}{\sqrt{\pi}} = \frac{80\sqrt{5}}{\sqrt{\pi}}$

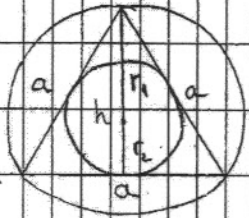
8. $a = 2$
 $\alpha = 60^\circ$

$H = \frac{a\sqrt{3}}{2}$
 $H = \frac{2\sqrt{3}}{2}$
 $H = \sqrt{3}$
 $a = 2r$
 $r = 1$
 $B = r^2\pi$
 $B = \pi$

$V = B \cdot H$
 $V = \pi\sqrt{3}$



9



a) $V = B_1 \cdot H$
 $V_1 = r_1^2 \pi$
 $V_2 = B_2 \cdot H$
 $B_2 = \frac{a^2 \sqrt{3}}{4}$
 $V_3 = B_3 \cdot H$
 $B_3 = r_2^2 \pi$
 $r_1 = \frac{2}{3} h$
 $r_2 = \frac{1}{3} h$
 $a = \frac{2h}{\sqrt{3}}$

$V_1 = \frac{4}{9} h^2 \pi \cdot H$
 $V_2 = \frac{\frac{4h^2}{3} \sqrt{3}}{4} \cdot H = \frac{h^2 \sqrt{3}}{3} H$
 $V_3 = \frac{1}{9} h^2 \pi H$
 $V_1 : V_2 : V_3 = \frac{4\pi}{9} : \frac{\sqrt{3}}{3} : \frac{\pi}{9} \quad | \cdot 9$
 $V_1 : V_2 : V_3 = 4\pi : 3\sqrt{3} : \pi$

b) $M_1 = 2r_1 \pi H$ $M_3 = 2r_2 \pi H$ $M_2 = 3aH$
 $M_1 = 2 \cdot \frac{2}{3} h \pi H$ $M_2 = \frac{2}{3} h \pi H$ $M_2 = 3 \cdot \frac{2h}{\sqrt{3}} H$
 $M_1 = \frac{4}{3} h \pi H$ $M_2 = \frac{6h}{\sqrt{3}} H$
 $M_2 = \frac{6h\sqrt{3}}{3} H$
 $M_2 = 2h\sqrt{3} H$

$M_1 : M_2 : M_3 = \frac{4}{3} h \pi H : 2h\sqrt{3} H : \frac{2}{3} h \pi H$
 $M_1 : M_2 : M_3 = \frac{4\pi}{3} : 2\sqrt{3} : \frac{2\pi}{3} \quad | \cdot 3$
 $M_1 : M_2 : M_3 = 4\pi : 6\sqrt{3} : 2\pi \quad | : 2$
 $M_1 : M_2 : M_3 = 2\pi : 3\sqrt{3} : \pi$

10. $R_1 = R + 1$ $P = 4R^2 \pi$
 $P_1 = P + 8\pi$ $4(R+1)^2 \pi = 4R^2 \pi + 8\pi$
 $(R^2 + 2R + 1)4\pi = 4\pi(R^2 + 2)$
 $R^2 + 2R + 1 = R^2 + 2$
 $2R = 1$
 $R = \frac{1}{2}$

$R_1 = \frac{1}{2} + 1$ $V_2 = \frac{4}{3} R_1^3 \pi$
 $R_1 = \frac{3}{2}$ $V_2 = \frac{4}{3} \cdot \frac{27}{8} \pi$
 $V_1 = \frac{4}{3} R^3 \pi$ $V_2 = \frac{9}{2} \pi$
 $V_1 = \frac{4}{3} \cdot \frac{1}{8} \pi$
 $V_1 = \frac{\pi}{6}$
 $V_2 - V_1 = \frac{9\pi}{2} - \frac{\pi}{6} = \frac{27\pi}{6} - \frac{\pi}{6} =$
 $= \frac{26\pi}{6} = \frac{13\pi}{3}$

2. Piramida i kuća

- Zapremina piramide baze B i visine H $V = \frac{1}{3} BH$

- Površina piramide baze B i visine H $P = B + M$

- Zapremina kupe (konusa) poluprečnika osnove r i visine H

$$V = \frac{1}{3} B H = \frac{r^2 \pi H}{3}$$

- Površina okrugla prave kupe poluprečnika osnove r i izvodnice s

$$M = rTs$$

- Površina okrugla prave kupe poluprečnika osnove r i izvodnice s

$$P = B + M = r^2 \pi + rTs = r(r\pi + Ts)$$

$$11. \quad M = 2S$$

$$M = \pi r s$$

$$\angle = 120^\circ$$

$$rTs = 2r^2 s$$

$$M = \frac{s^2 \pi}{2}$$

$$s = 2r$$

$$M = \frac{r^2 \pi}{2}$$

$$r = \frac{s}{2}$$

12. a)

$$B = \left(\frac{a}{2}\right)^2 \pi$$

$$M = \frac{a}{2} \pi a$$

$$P = B + M$$

$$V = \frac{1}{3} B h$$

$$B = \frac{a^2}{4} \pi$$

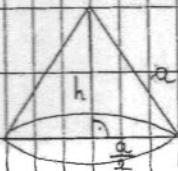
$$M = \frac{a^2}{2} \pi$$

$$P = \frac{a^2 + 2a^2}{4} \pi$$

$$V = \frac{1}{3} \frac{a^2 \pi}{4} \frac{a\sqrt{3}}{2}$$

$$P = \frac{3a^2 \pi}{4}$$

$$V = \frac{a^3 \pi \sqrt{3}}{24}$$



b)



$$B = h^2 \pi$$

$$M = h \pi a$$

$$V = a \cdot \frac{1}{3} B \cdot \frac{a}{2}$$

$$B = \left(\frac{a\sqrt{3}}{2}\right)^2 \pi$$

$$M = \frac{a^2 \sqrt{3}}{2} \pi$$

$$V = \frac{3a^3 \pi}{4} \cdot \frac{1}{3}$$

$$B = \frac{3a^2}{4} \pi$$

$$P = 2M$$

$$V = \frac{a^3 \pi}{4}$$

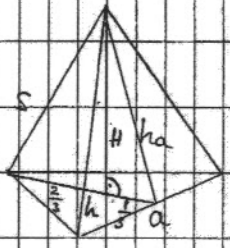
$$P = a^2 \sqrt{3} \pi$$

13. $a = \sqrt{3}$
 $V = 3$

$V = \frac{1}{3} B H$
 $V = \frac{1}{3} \frac{a^2 \sqrt{3}}{4} \cdot H$

$H = \frac{12V}{a^2 \sqrt{3}}$
 $H = \frac{36}{3\sqrt{3}} = \frac{36\sqrt{3}}{9}$
 $H = 4\sqrt{3}$

$B = \frac{a^2 \sqrt{3}}{4}$
 $B = \frac{3\sqrt{3}}{4}$



$h_a^2 = H^2 + (\frac{1}{2}a)^2$
 $h_a^2 = 48 + (\frac{a\sqrt{3}}{2})^2$
 $h_a^2 = 48 + \frac{9}{36} = 48 + \frac{1}{4}$

$U = 3 \frac{a \cdot h_a}{2}$
 $U = \frac{3\sqrt{3} \sqrt{193}}{4}$

$h_a^2 = \frac{193}{4}$
 $h_a = \frac{\sqrt{193}}{2}$

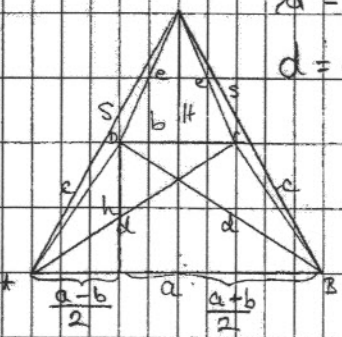
$P = B + U$
 $P = \frac{3\sqrt{3}}{4} \cdot (1 + \sqrt{193})$

14. $a = 5$
 $b = 3$
 $c = 7$
 $s = 10$

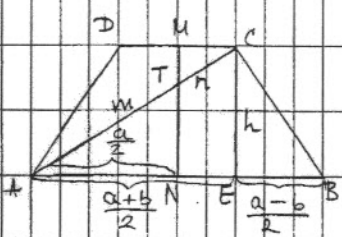
a) $h^2 = c^2 - (\frac{a-b}{2})^2$
 $h^2 = 49 - 1$
 $h = \sqrt{48} = 4\sqrt{3}$
 $d^2 = h^2 + (\frac{a+b}{2})^2$
 $d^2 = 48 + 16$
 $d^2 = 64$
 $d = 8$

$s^2 = u^2 + h^2$
 $h^2 = 100 - 25$
 $h^2 = 75$
 $h = 5\sqrt{3}$

$V = \frac{1}{3} B h$
 $V = \frac{1}{3} \frac{a+b}{2} \cdot h \cdot h$
 $V = \frac{1}{3} 16\sqrt{3} \cdot 5\sqrt{3}$
 $V = 80$



b) $n = d - u = 8 - 5$

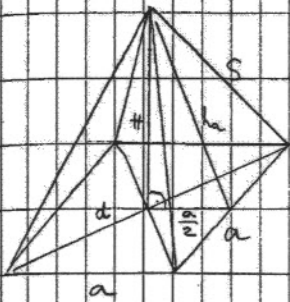


$\triangle CAE \sim \triangle NTA$
 $AE : AN = AC : AT$
 $\frac{a+b}{2} : \frac{a}{2} = d : u$
 $h : \frac{5}{2} = 8 : u$
 $h \cdot u = 20$
 $u = 5$

$n = 3$
 $e^2 = h^2 + u^2$
 $e^2 = 75 + 9$
 $e^2 = 84$
 $e = 2\sqrt{21}$

15. $P_d = 12$

$O = 8$



a) $O = 4a$

$a = 2$

$d = a\sqrt{2}$

$d = 2\sqrt{2}$

$P_d = \frac{d \cdot H}{2}$

$H = \frac{2P_d}{d}$

$H = \frac{2 \cdot 12}{2\sqrt{2}}$

$H = 6\sqrt{2}$

$ka^2 = H^2 + \left(\frac{a}{2}\right)^2$

$ka^2 = 72 + 1$

$ka = \sqrt{73}$

$B = a^2$

$B = 4$

$M = \frac{1}{2} \cdot \frac{a \cdot ka}{2}$

$M = 4\sqrt{73}$

$P = B + M$

$P = 4 + 4\sqrt{73}$

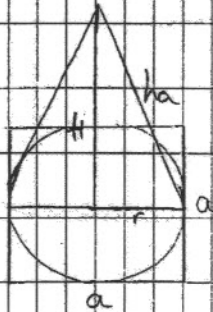
$P = 4(1 + \sqrt{73})$

$V = \frac{1}{3} B H$

$V = \frac{1}{3} \cdot 4 \cdot 6\sqrt{2}$

$V = 8\sqrt{2}$

b)



$r = \frac{a}{2}$

$r = 1$

$B = r^2 \pi$

$B = \pi$

$M = r \pi h a$

$M = \sqrt{3} \pi$

$P = B + M$

$P = \pi(1 + \sqrt{3})$

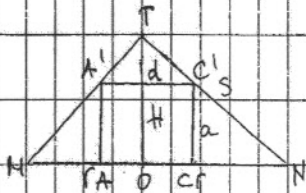
$V = \frac{1}{3} B H$

$V = \frac{1}{3} \pi \cdot 1^2 \cdot 2\sqrt{2}$

$V = 2\sqrt{2} \pi$

16. $H = r\sqrt{2}$

$d = a\sqrt{2}$



$\Delta TON \sim \Delta C'CN$

$TO : C'C = ON : CN$

$H : a = r : \left(r - \frac{d}{2}\right)$

$r\sqrt{2} : a = r : \left(r - \frac{a\sqrt{2}}{2}\right)$

$r^2\sqrt{2} - \frac{ra\sqrt{2}}{2} = ra$

$r^2\sqrt{2} - ra = ra$

$2ra = r^2\sqrt{2}$

$a = \frac{r^2\sqrt{2}}{2r} = \frac{r\sqrt{2}}{2}$

$V_1 = \frac{1}{3} B_1 H$

$V_1 = \frac{1}{3} r^2 \pi r\sqrt{2}$

$V_1 = \frac{1}{3} r^3 \pi \sqrt{2}$

$V_1 = \frac{r^3 \pi \sqrt{2}}{3}$

$V_2 = a^3$

$V_2 = \left(\frac{r\sqrt{2}}{2}\right)^3$

$V_2 = \frac{r^3 \sqrt{2}}{4}$

$V_2 = \frac{r^3 \sqrt{2}}{4}$

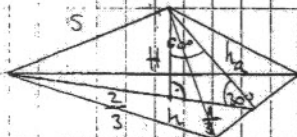
$\frac{V_1}{V_2} = \frac{\frac{r^3 \pi \sqrt{2}}{3}}{\frac{r^3 \sqrt{2}}{4}}$

$\frac{V_1}{V_2} = \frac{4\pi}{3}$

$\frac{V_1}{V_2} = \frac{4\pi}{3}$

17. $a = 10$

$\alpha = 30^\circ$



a) $B = \frac{a^2 \sqrt{3}}{4}$

$B = \frac{100\sqrt{3}}{4}$

$B = 25\sqrt{3}$

$\frac{1}{3} h = \frac{2H\sqrt{3}}{2}$

$\frac{a\sqrt{3}}{6} = \frac{2H\sqrt{3}}{2}$

$6H = a$

$H = \frac{10}{6} = \frac{5}{3}$

$ka = 2H$

$ka = \frac{10}{3}$

$$M = 3 \cdot \frac{a \cdot na}{2} = 3 \cdot \frac{10 \cdot \frac{10}{3}}{2}$$

$$M = 50$$

$$P = B + M$$

$$P = 25\sqrt{3} + 50$$

$$P = 25(\sqrt{3} + 2)$$

$$b) S^2 = H^2 + \left(\frac{2}{3}h\right)^2$$

$$S^2 = H^2 + \left(\frac{2}{3} \cdot \frac{a\sqrt{3}}{2}\right)^2$$

$$S^2 = \frac{25}{9} + \frac{300}{9}$$

$$S^2 = \frac{325}{9}$$

$$S = \frac{5\sqrt{13}}{3}$$

8.3 Zarubljena piramida i zarubljena kupa

- Zapremina zarubljene piramide čije su baze B_1 i B_2 , a visina H

$$V = \frac{1}{3} H (B_1 + \sqrt{B_1 B_2} + B_2)$$

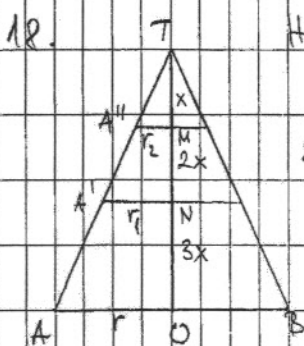
- Površina zarubljene piramide čije su baze B_1 i B_2 , a osovina M

$$P = B_1 + B_2 + M$$

- Zapremina zarubljene kupa poluprečnika osnova R i r i visine H

$$V = \frac{1}{3} H (B_1 + \sqrt{B_1 B_2} + B_2) = \frac{\pi H}{3} (R^2 + Rr + r^2)$$

- Površina zarubljene kupa poluprečnika osnova R i r i izvodnice l

$$P = B_1 + B_2 + M = R^2\pi + r^2\pi + (R+r)\pi l = \pi(R^2 + r^2 + Rl + rl)$$


$$H = 6x$$

$$Gx : r = x : r_2 \quad V_1 = \frac{1}{3} H \pi (r^2 + r_1 r_2 + r_2^2)$$

$$Gx : r = 3x : r_1$$

$$Gx r_2 = r x$$

$$V_1 = \frac{1}{3} H \pi \left(\frac{r^2}{4} + \frac{r^2}{12} + \frac{r^2}{36} \right)$$

$$2 Gx r_1 = 3x r$$

$$r = 6r_2$$

$$V_1 = \frac{1}{3} H \pi \frac{13}{36} r^2$$

$$r = 2r_1$$

$$r_2 = \frac{r}{6}$$

$$V_1 = \frac{1}{3} \pi 2x \cdot \frac{13 r^2}{36} = \frac{13}{18} \pi x r^2$$

$$r_1 = \frac{r}{2}$$

$$V_1 = \frac{1}{3} \pi x r^2 \frac{13}{36}$$

$$V = \frac{1}{3} r^2 \pi H$$

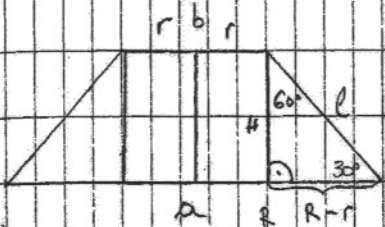
$$\frac{V_1}{V} = \frac{\frac{1}{3} \pi x r^2 \frac{13}{36}}{\frac{1}{3} r^2 \pi 6x}$$

$$V = \frac{1}{3} r^2 \pi Gx$$

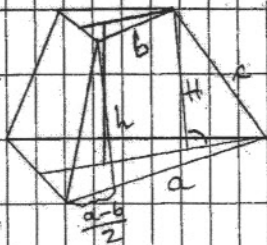
$$\frac{V_1}{V} = \frac{13}{108}$$

$$\frac{V_1}{V_2} = \frac{13}{108}$$

19. $\alpha = 30^\circ$ $a = 2R$ $P = \frac{a+b}{2} \cdot H$ $H = \frac{l}{2}$ $M = (R+r)\pi l$
 $P = 5$ $b = 2r$ $P = \frac{2R+2r}{2} \cdot H$ $(R+r)\frac{l}{2} = 5$ $M = 10\pi$
 $(R+r)H = 5$ $(R+r)l = 10$



20. $a = 9$ a) $c^2 = h_1^2 + \left(\frac{a-b}{2}\right)^2$ $B_1 = \frac{a^2\sqrt{3}}{4}$
 $b = 3$ $h_1^2 = 25 - 9$ $B_1 = \frac{81\sqrt{3}}{4}$
 $c = 5$ $h_1^2 = 16$ $B_2 = \frac{b^2\sqrt{3}}{4}$
 $h_1 = 4$ $B_2 = \frac{9\sqrt{3}}{4}$
 $M = 3 \frac{a+b}{2} \cdot H$
 $M = 72$



$P = B_1 + B_2 + M$
 $P = \frac{81\sqrt{3}}{4} + \frac{9\sqrt{3}}{4} + 72$
 $P = \frac{90\sqrt{3}}{4} + 72$
 $P = \frac{45\sqrt{3}}{2} + 72$
 $c^2 = H^2 + \left(\frac{2}{3}h_1 - \frac{2}{3}h_2\right)^2$
 $H^2 = c^2 - \left(\frac{2}{3} \frac{81\sqrt{3}}{2}\right)^2$
 $H^2 = 25 - 12$
 $H^2 = 13$
 $H = \sqrt{13}$

$V = \frac{1}{3} (B_1 + \sqrt{B_1 B_2} + B_2) H$
 $V = \frac{1}{3} \left(\frac{81\sqrt{3}}{4} + \sqrt{\frac{81\sqrt{3}}{4} \cdot \frac{9\sqrt{3}}{4}} + \frac{9\sqrt{3}}{4} \right) \cdot \sqrt{13}$
 $V = \frac{\sqrt{13}}{3} \cdot \frac{81\sqrt{3} + 27\sqrt{3} + 9\sqrt{3}}{4}$
 $V = \frac{39\sqrt{39}}{4}$

b) $R = \frac{2}{3}h_1 = 3\sqrt{3}$ $P = \pi(R^2 - r^2 + Rl + rl)$ $V = \frac{1}{3}\pi H(R^2 - Rr + r^2)$
 $r = \frac{2}{3}h_2 = \sqrt{3}$ $P = \pi(27 + 3 + 15\sqrt{3} + 5\sqrt{3})$ $V = \frac{1}{3}\pi\sqrt{13}(27 + 9 + 3)$
 $l = c = 5$ $P = \pi(30 + 20\sqrt{3})$ $V = \frac{1}{3}\pi\sqrt{13} \cdot 39\sqrt{13}$
 $P = 10\pi(3 + 2\sqrt{3})$ $V = 13\pi\sqrt{13}$

8.4. Zadaci za vežbu

1. $a = 30$

$r = 10$

$H = h$

$\triangle ABC \sim \triangle FCE$

$\frac{a}{2} : b = r : h - r$

$15 : b = 10 : h - 10$

$15h - 150 = 10b$

$b = \frac{3(3h - 30)}{2}$

$b = \frac{3h - 30}{2}$

$b^2 = h^2 + \left(\frac{a}{2}\right)^2$

$\frac{9h^2 - 180h + 900}{4} = h^2 + 225$

$9h^2 - 180h + 900 = 4h^2 + 900$

$5h^2 - 180h = 0$

$5h(h - 36) = 0$

$h - 36 = 0$

$h = H = 36$

$V = Bh$

$V = \frac{a \cdot h}{2} \cdot H$

$V = \frac{30 \cdot 15 \cdot 36}{2}$

$V = 19440$

2. $N_1 = P$

$2r\pi h_1 = 2r\pi(r+h)$

$h_1 = r+h$

$V = r^2\pi h$

$V_1 = r^2\pi h_1$

$V_1 = r^2\pi(r+h)$

$V_1 = r^3\pi + r^2\pi h$

$V_1 + V = r^3\pi + r^2\pi h + r^2\pi h$

$V_1 + V = r^3\pi$

$V_1 = V + r^3\pi$

3. $a = 10$

$b = 2$

$P = 90$

$P = \frac{a+b}{2} \cdot h$

$h = \frac{2P}{a+b}$

$h = \frac{180}{12}$

$h = 15$

$d^2 = c^2 + (a-b)^2$

$d^2 = 225 + 64$

$d^2 = 289$

$d = 17$

$P_1 = c^2\pi + 2c\pi b$

$P_1 = 225\pi + 60\pi$

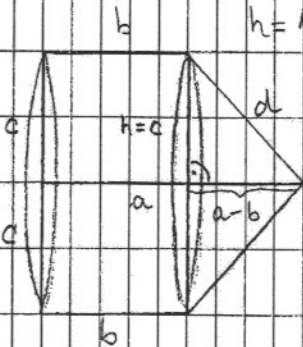
$P_1 = 285\pi$

$P_2 = c\pi d$

$P_2 = 225\pi$

$P = P_1 + P_2$

$P = 540\pi$



$V_1 = c^2\pi b$

$V_1 = 225\pi \cdot 2$

$V_1 = 450\pi$

$V_2 = \frac{1}{3}c^2\pi \cdot (a-b)$

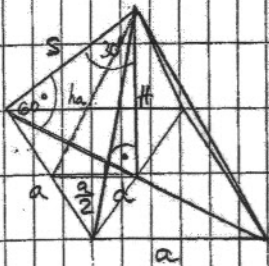
$V_2 = \frac{1}{3} \cdot 225\pi \cdot 8$

$V_2 = 600\pi$

$V = V_1 + V_2$

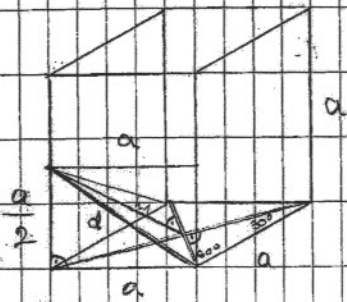
$V = 1050\pi$

$$\begin{array}{llllll}
 4. \quad \alpha = 2 & d = a\sqrt{2} & H = \frac{d\sqrt{3}}{2} & h_a^2 = H^2 + \left(\frac{a}{2}\right)^2 & V = \frac{1}{3} B H \\
 \alpha = 60^\circ & d = 2\sqrt{2} & H = \frac{2\sqrt{6}}{2} & h_a^2 = 6 + 1 & V = \frac{1}{3} 4 \cdot \sqrt{6} \\
 & S = d & & h_a^2 = 7 & V = \frac{4\sqrt{6}}{3} \\
 & S = 2\sqrt{2} & H = \sqrt{6} & h_a = \sqrt{7} &
 \end{array}$$



$$\begin{array}{lll}
 B = a^2 & U = 2aha & P = B + U \\
 B = 4 & U = 4\sqrt{7} & P = 4(1 + \sqrt{7})
 \end{array}$$

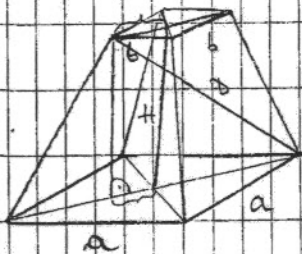
$$\begin{array}{llllll}
 5. \quad \alpha = 2 & d_2 = a & H = a & \frac{d_1}{2} = \frac{a\sqrt{3}}{2} & B = \frac{d_1 d_2}{4} & d^2 = a^2 + \left(\frac{a}{2}\right)^2 \\
 \alpha = 60^\circ & d_2 = 2 & h = \frac{H}{2} & d_1 = a\sqrt{3} & B = \frac{4\sqrt{3}}{4} & d^2 = 4 + 1 \\
 & & h = 1 & d_1 = 2\sqrt{3} & B = \sqrt{3} & d^2 = 5 \\
 & & & & & d = \sqrt{5}
 \end{array}$$



$$\begin{array}{lll}
 P_1 = P_2 = \frac{a \cdot \frac{a}{2}}{2} & h_1^2 = d^2 - \left(\frac{d_2}{2}\right)^2 & P_3 = \frac{d_2 \cdot h_1}{2} \\
 P_1 = P_2 = 1 & h_1^2 = 5 - 1 & P_3 = \frac{2 \cdot 2}{2} \\
 & h_1^2 = 4 & P_3 = 2 \\
 & h_1 = 2 &
 \end{array}$$

$$\begin{array}{ll}
 U = P_1 + P_2 + P_3 & P = B + U \\
 U = 4 & P = \sqrt{3} + 4
 \end{array}$$

$$\begin{array}{llll}
 6. \quad B_1 = 4 & B_1 = a^2 & V = \frac{1}{3} H (B_1 + \sqrt{B_1 B_2} + B_2) & D^2 = H^2 + \left(\frac{da}{2} + \frac{db}{2}\right)^2 \\
 B_2 = 1 & a = 2 & 21 = H(4 + 2 + 1) & D^2 = H^2 + \left(\frac{3\sqrt{2}}{2}\right)^2 \\
 V = 4 & B_2 = b^2 & H = 3 & D^2 = 9 + \frac{18}{4} \\
 & b = 1 & d_a = a\sqrt{2} & D^2 = \frac{54}{4} \\
 & & d_a = 2\sqrt{2} & D = \frac{3\sqrt{6}}{2} \\
 & & d_b = b\sqrt{2} & \\
 & & d_b = \sqrt{2} &
 \end{array}$$



7. $S = 5$

$$S^2 = H^2 + (R-r)^2$$

$$M = (R+r)\pi S$$

$R = 5$

$$H^2 = 25 - 16$$

$$M = 30\pi$$

$r = 1$

$$H^2 = 9$$

$$M = 4\pi$$

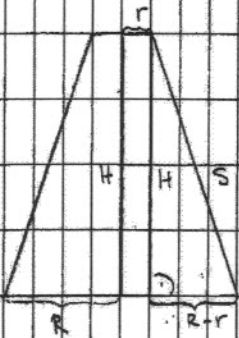
$$H = 3$$

$$M_1 = 2R_1\pi H$$

$$R_1 = \frac{M_1}{2\pi H}$$

$$R_1 = \frac{30\pi}{2\pi \cdot 3}$$

$$R_1 = 5$$



8. $\triangle ABC \sim \triangle EDC$

$$CE:CA = ED:AB$$

$$H^2 = h^2 - \left(\frac{1}{3}h\right)^2$$

$$H_1 = a_1 \frac{\sqrt{2}}{\sqrt{3}}$$

$$\frac{2}{3}h:h = a_1:\frac{a}{2}$$

$$H^2 = h^2 - \frac{1}{9}h^2$$

$$H_1 = \frac{a\sqrt{2}}{3\sqrt{3}}$$

$$a_1 h = \frac{2}{3}h \frac{a}{2}$$

$$H^2 = \frac{8}{9}h^2$$

$$V = \frac{1}{3} \frac{a^2 \sqrt{3}}{4} \frac{a\sqrt{2}}{\sqrt{3}}$$

$$a_1 = \frac{a}{3}$$

$$H^2 = \frac{8}{9} \left(\frac{a\sqrt{3}}{2}\right)^2$$

$$V = \frac{a^3 \sqrt{2}}{12}$$

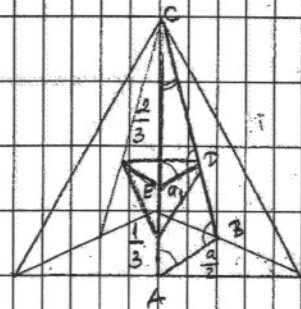
$$H^2 = \frac{2}{3} \frac{a^2 \cdot 3}{4}$$

$$V_1 = \frac{1}{3} \frac{a^2 \sqrt{3}}{9} \frac{a\sqrt{2}}{\sqrt{3}}$$

$$H^2 = \frac{2}{3} a^2$$

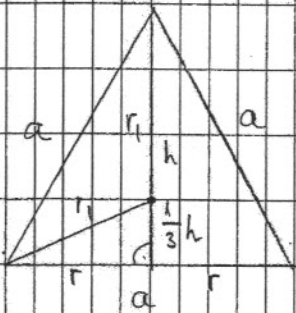
$$V_1 = \frac{a^3 \sqrt{2}}{324}$$

$$H = a \frac{\sqrt{2}}{\sqrt{3}}$$



$$\frac{V_1}{V} = \frac{\frac{a^3 \sqrt{2}}{324}}{\frac{a^3 \sqrt{2}}{12}} = \frac{324}{12} = 27:1$$

9.



$$r = \frac{a}{2}$$

$$r_1^2 = r^2 + \left(\frac{1}{3}h\right)^2$$

$$V_1 = \frac{4}{3} \frac{a^3}{9\sqrt{3}} \pi$$

$$V = \frac{1}{3} \left(\frac{a^2}{4}\right) \pi \cdot \frac{a\sqrt{3}}{2}$$

$$r_1^2 = \left(\frac{a}{2}\right)^2 + \frac{1}{9} \left(\frac{a\sqrt{3}}{2}\right)^2$$

$$V_1 = \frac{4a^3 \pi}{9\sqrt{3}}$$

$$V = \frac{1}{3} \frac{a^2}{4} \pi \frac{a\sqrt{3}}{2}$$

$$r_1^2 = \frac{a^2}{4} + \frac{1}{9} \frac{3a^2}{4}$$

$$V = \frac{a^3 \sqrt{3} \pi}{24}$$

$$V = \frac{a^3 \sqrt{3} \pi}{24}$$

$$r_1^2 = \frac{a^2}{4} + \frac{3}{9} \frac{a^2}{4}$$

$$V_1 = \frac{4a^3 \pi}{9\sqrt{3}}$$

$$r_1^2 = \frac{a^2}{3}$$

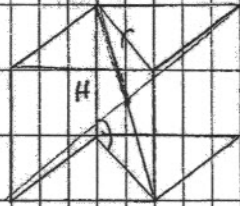
$$\frac{V}{V_1} = \frac{27}{96}$$

$$r_1 = \frac{a}{\sqrt{3}}$$

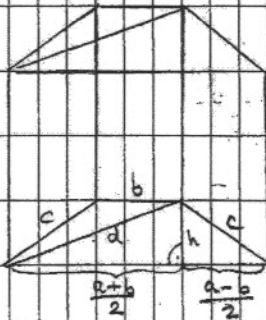
$$V:V_1 = 9:32$$

10. $\alpha = 4$ $P = 4r^2\pi$ $D = 2r$ $D^2 = 4^2 + d^2$ $P_d = 4 \cdot d$
 $P = 36\pi$ $r^2 = \frac{P}{4\pi}$ $D = 6$ $4^2 = 36 - d^2$ $P_d = 8\sqrt{2}$
 $r^2 = \frac{36\pi}{4\pi}$ $d = 0\sqrt{2}$ $4^2 = 4$
 $d = 4\sqrt{2}$ $4 = 2$

$r^2 = 9$
 $r = 3$



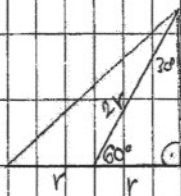
11. $c = 13$ $c^2 = h^2 + \left(\frac{a-b}{2}\right)^2$ $d^2 = h^2 + \left(\frac{a+b}{2}\right)^2$
 $\alpha = 21$ $\left(\frac{21-b}{2}\right)^2 = 169 - 144$ $d^2 = 144 + 256$
 $h = 12$ $441 - 42b + b^2 = 25$ $d^2 = 400$
 $P = 906$ $b^2 - 42b + 341 = 0$ $d = 20$



$b^2 - 42b + 341 = 0$
 $b_{1,2} = \frac{42 \pm \sqrt{1764 - 1364}}{2} \rightarrow b_1 = 31 \rightarrow a > b$
 $\rightarrow b_2 = 11$

$P = 2B + 4h$ $P_d = d \cdot h$
 $P = 2 \cdot \frac{a+b}{2} \cdot h + (a-b+2c) \cdot h$ $P_d = 20 \cdot 9$
 $906 = 384 + 58h$ $P_d = 180$
 $58h = 522$
 $h = 9$

12. $\alpha = 60^\circ$ $h = \frac{2+\sqrt{3}}{2}$ $V = \frac{1}{3} r^2 \pi h$
 $2r = 2$ $h = \sqrt{3}$ $V = \frac{1}{3} 1 \pi \sqrt{3}$
 $r = 1$ $V = \frac{\sqrt{3}\pi}{3}$



13. $a = 2$

$r = \frac{a}{2}$

$s^2 = H^2 + r^2$

$B = a^2$

$P = (1 + \sqrt{3})\pi$

$r = 1$

$H^2 = 3 - 1$

$B = 4$

$P = r^2\pi + rTs$

$H^2 = 2$

$H = 2\sqrt{2}$

$(1 + \sqrt{3})\pi = \pi(1 + s)$

$H = 6\sqrt{2}$

$M = 4\sqrt{3}$

$1 + s = 1 + \sqrt{3}$

$s = \sqrt{3}$

$D = B + M$

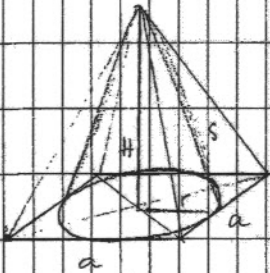
$V = \frac{1}{3}BH$

$P = 4(1 + \sqrt{3})$

$V = \frac{1}{3} \cdot 4 \cdot 2\sqrt{2}$

$\frac{2}{3} \cdot 3\sqrt{3}$

$V = 8\sqrt{2}$



14. $P = 32$

$P = 6a^2$

$r = \frac{D}{2}$

$V = \frac{4}{3}r^3\pi$

$V_3 = \frac{V_2}{6}$

$a^2 = \frac{32}{6}$

$r = 2$

$V_1 = \frac{4}{3} \cdot 8 \cdot \pi$

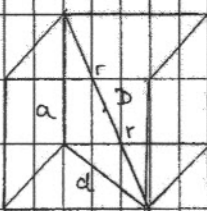
$V_3 = \frac{32(3\pi - 2\sqrt{3})}{9}$

$a^2 = \frac{16}{3}$

$V = a^3$

$V_1 = \frac{32\pi}{3}$

$V_3 = \frac{32(3\pi - 2\sqrt{3})}{9}$



$a = \frac{4\sqrt{3}}{3}$

$V = \frac{64 \cdot 2\sqrt{3}}{27 \cdot 9}$

$V_2 = V_1 - V$

$V_3 = \frac{32(3\pi - 2\sqrt{3})}{54}$

$D = a\sqrt{3}$

$V = \frac{64\sqrt{3}}{9}$

$V_2 = \frac{96\pi - 64\sqrt{3}}{9}$

$V_3 = \frac{16}{27}(3\pi - 2\sqrt{3})$

$D = 4$

$V_2 = \frac{32(3\pi - 2\sqrt{3})}{9}$

15. $a = 4$

$a^2 = \left(\frac{d_1}{2}\right)^2 + \left(\frac{d_2}{2}\right)^2$

$P = \frac{d_1 d_2}{2}$

$V = B \cdot h$

$d_1 = 2\sqrt{7}$

$\left(\frac{d_2}{2}\right)^2 = 16 - 7$

$P = a \cdot h$

$V = \frac{3a^2\sqrt{3}}{2} \cdot \frac{3\sqrt{7}}{2}$

$\left(\frac{d_2}{2}\right)^2 = 9$

$h = \frac{d_1 d_2}{2a}$

$V = \frac{3 \cdot 16\sqrt{3}}{2} \cdot \frac{3\sqrt{7}}{2}$

$\frac{d_2}{2} = 3$

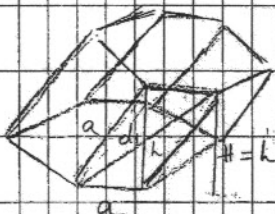
$h = \frac{2\sqrt{7} \cdot 6}{2 \cdot 4}$

$V = 36\sqrt{21}$

$d_2 = 6$

$h = \frac{3\sqrt{7}}{2}$

$h = 4$



16. $\alpha = 60^\circ$

$h = \frac{a\sqrt{3}}{2}$

$V = r^2 \pi h$

$V = 8\pi\sqrt{3}$

$h = \frac{2r\sqrt{3}}{2}$

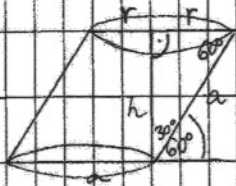
$V = r^2 \pi r\sqrt{3}$

$8\pi\sqrt{3} = r^3 \pi\sqrt{3}$

$h = r\sqrt{3}$

$r^3 = 8$

$r = 2$



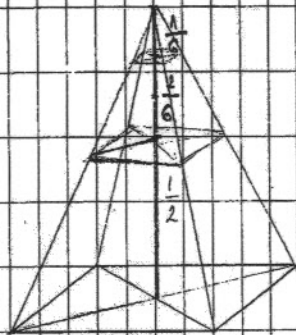
17.

$\frac{d}{2} \cdot h = \frac{d_1}{2} \cdot \frac{h}{2}$

$a : a_1 = d : d_1$

$\frac{d}{2} : h = \frac{d_2}{2} : \frac{1}{6}h$

$a : a_2 = d : d_2$



$\frac{dh}{4} = \frac{d_1 h_1}{2}$

$ad_1 = a_1 d$

$\frac{dh}{12} = \frac{d_2 h}{2}$

$d_2 a = da_2$

$a_1 = \frac{a}{2}$

$a_1 = \frac{a}{2}$

$d_2 = \frac{a}{6}$

$a_2 = \frac{a}{6}$

$B = a^2$

$V = \frac{1}{3} B h$

$V_1 = \frac{1}{3} \frac{1}{3} h \left(\frac{a^2}{4} + \sqrt{\frac{a^2}{4} \frac{a^2}{36}} + \frac{a^2}{36} \right)$

$B_1 = a_1^2 = \frac{a^2}{4}$

$V = \frac{1}{3} a^2 h$

$V_1 = \frac{1}{9} h \left(\frac{10a^2}{36} + \frac{a^2}{12} \right)$

$B_2 = a_2^2 = \frac{a^2}{36}$

$V_1 = \frac{1}{9} h \left(\frac{13a^2}{36} \right)$

$\frac{V}{V_1} = \frac{\frac{1}{3} a^2 h}{\frac{1}{9} \frac{13a^2}{36} h}$

$V : V_1 = 108 : 13$

18. $\alpha = 30^\circ$

$N = (R+r)\pi s$

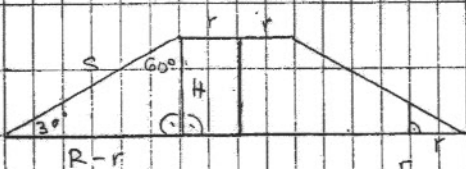
$R-r = \frac{s\sqrt{3}}{2}$

$N = 10\pi$

$(R+r)\pi s = 10\pi$

$s = \frac{2(R-r)}{\sqrt{3}}$

$(R+r) \cdot s = 10$



$$19. \quad r^2 = \left(\frac{a}{2}\right)^2 - \left(\frac{1}{3} \frac{a\sqrt{3}}{2}\right)^2 \quad r^2 = \frac{2a^2}{12}$$

$$r^2 = \frac{a^2}{4} - \frac{3a^2}{36 \cdot 12} \quad r = \frac{a}{6}$$

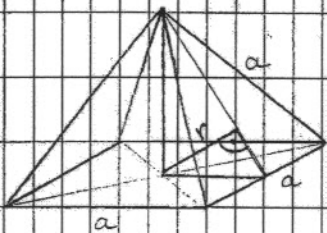
$$V = \frac{1}{3} r^2 H$$

$$V = \frac{1}{3} \cdot \frac{a^2}{6 \cdot 6} \cdot H$$

$$V = \frac{2a^2 \sqrt{3} \pi}{9 \cdot 36} \cdot \frac{\sqrt{6}}{\sqrt{6}}$$

$$V = \frac{2a^2 \sqrt{6} \pi}{342}$$

$$V = \frac{a^2 \sqrt{6} \pi}{27}$$



$$21. \quad x = \sqrt{2}$$

$$ha^2 = \left(\frac{a}{2}\right)^2 + x^2$$

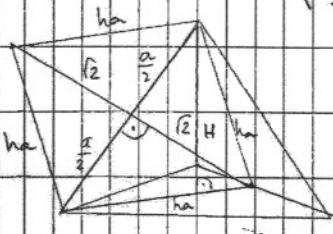
$$\left(\frac{a\sqrt{3}}{2}\right)^2 = \frac{a^2}{4} + 2$$

$$\frac{3a^2}{4} = \frac{a^2}{4} + 2$$

$$\frac{2a^2}{4} = 2$$

$$a^2 = 4$$

$$a = 2$$



$$H^2 = ha^2 - \left(\frac{1}{2} ha\right)^2$$

$$H^2 = ha^2 - \frac{1}{4} ha^2$$

$$H^2 = \frac{28}{3} \frac{8a^2}{4}$$

$$H^2 = \frac{2}{3} a^2$$

$$H = \frac{2\sqrt{2}}{\sqrt{3}}$$

$$V = \frac{1}{3} B H$$

$$V = \frac{1}{3} \cdot \frac{a^2 \sqrt{3}}{4} \cdot H$$

$$V = \frac{1}{3} \cdot \frac{4\sqrt{3}}{4} \cdot \frac{2\sqrt{2}}{\sqrt{3}}$$

$$V = \frac{2\sqrt{2}}{3}$$