b) $\vec{AB_1} \cdot \vec{A_1C_1} = (1,0,1)\cdot(1,1,0) = 1$

c) $E = \left(\frac{1+0}{2}, \frac{1+0}{2}, \frac{0+0}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$

$\vec{B_1T} : \vec{TE} = 2 : 1$

$\vec{r_T} = \frac{\vec{r_{B_1}} + 2\vec{r_E}}{1+2}$

$\vec{r_T} = \frac{(1,0,1)+(1,1,0)}{3}$

$\vec{r_T} = \frac{(2,1,1)}{3}$

$T = \left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$

6. Analitička geometrija u ravni

6.1. Tačka i prava

- Svaka tačka T u ravni se zadaje svojim koordinatama $T(x_1, y_1)$
- Vektor koji spaja tačke $A(x_1, y_1)$ i $B(x_2, y_2)$ je:

$\vec{AB} = (x_2 - x_1, y_2 - y_1)$

- Rastojanje tačaka A i B je:

$d = AB = |\vec{AB}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

- Središte duži AB je $S\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
- Jednačina prave:

Opšti oblik: $y = kx + n$

$k = tg\,\alpha$ - koeficijent pravca

n - odsečak na y-osi

- Specijalni slučajevi pravih:

Jednačina prave normalne na x-osu $x = m$

Jednačina prave normalne na y-osu $y = n$

- Prava kroz tačku $A(x_1, y_1)$ sa koeficijentom pravca k:

$y - y_1 = k(x - x_1)$

Prava kroz tačke $A(x_1, y_1)$ i $B(x_2, y_2)$

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1} \Rightarrow k=\frac{y_2-y_1}{x_2-x_1}$$

Odnos pravih $y=kx+n$ i $y=k_1x+n_1$

1) prave su paralelne ako je $k=k_1$

2) prave su normalne ako je $k_1=-\frac{1}{k}$.

Presek pravih je tačka čije koordinate su rešenja sistema koji čine jednačine datih pravih.

Ugao φ između pravih $y=k_1x+n_1$ i $y=k_2x+n_2$ se nalazi iz veze:

$$\operatorname{tg}\varphi=\frac{k_2-k_1}{1+k_1k_2}$$

1. $3x-y+1=0 \mid 2$

$x+2y+5=0$

$6x-2y+2=0$

$x+2y+5=0$

$7x+7=0$

$x=-1$

$y=-2$

$2x-5y+6=0$

$5y=2x+6$

$y=\frac{2}{5}x+\frac{6}{5}$

$y-(-2)=\frac{2}{5}(x-(-1))$

$y+2=\frac{2}{5}(x+1)$

2. $A(4,2)$

$B(1,-7)$

$p: \frac{y-2}{-7-2}=\frac{x-4}{1-4}$

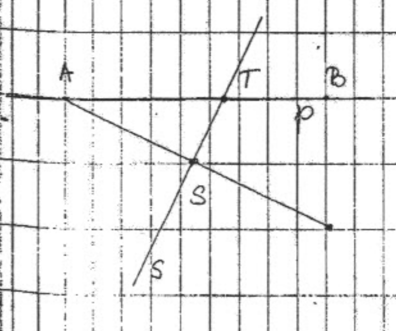
$\frac{y-2}{\cancel{-9}_{3}}=\frac{x-4}{\cancel{-3}_{1}}$

$y-2=3(x-4)$

$AO: k=\frac{2-0}{4-0}=\frac{1}{2}$

$s: k_1=-\frac{1}{k}=-2$

$S\left(\frac{4+0}{2}, \frac{2+0}{2}\right)=(2,1)$



$s: y-1=-2(x-2)$

$p: y-2=3(x-4)$

$y-1=-2x+4$

$y-2=3x-12$

$y=-2x+5 \mid (-1)$

$y=3x-10$

$-\cancel{y} = 2x - 5$

$\cancel{y} = 3x - 10$

$5x - 15 = 0$

$5x = 15$

$x = 3$

$y = -1 \qquad T(3,-1)$

3. $A(-3,-1)$ $B(1,1)$ $C(-2,3)$

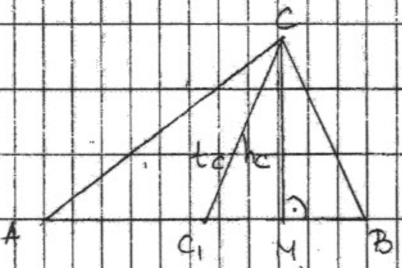
$AB: \frac{y+1}{1+1} = \frac{x+3}{1+3}$

$\frac{y+1}{\cancel{2}} = \frac{x+3}{\cancel{4}\,2}$

$y+1 = \frac{1}{2}(x+3)$

$y = \frac{1}{2}x + \frac{1}{2}$

$k_{CH} = -2$



$CH(h): y-3 = -2(x+2)$

$y-3 = -2x-4$

$y = -2x-1$

$C_1\left(\frac{-3+1}{2}, \frac{-1+1}{2}\right)$

$C_1(-1, 0)$

$CC_1(t_c): \frac{y-3}{0-3} = \frac{x+2}{-1+2}$

$\frac{y-3}{-3} = \frac{x+2}{1}$

$y-3 = -3(x+2)$

$y-3 = -3x-6$

$y = -3x-3$

$y = \frac{1}{2}x + \frac{1}{2} \quad | \cdot 4$

$y = -2x-1$

$4y = 2x+2$

$y = -2x-1$

$5y = 1$

$y = \frac{1}{5}$

$x = -\frac{3}{5}$

$H\left(-\frac{3}{5}, \frac{1}{5}\right)$

$h = \sqrt{\left(\frac{1}{5} - \frac{15}{5}\right)^2 + \left(-\frac{3}{5} + \frac{10}{5}\right)^2}$

$h = \sqrt{\frac{196}{25} + \frac{49}{25}}$

$h = \frac{\sqrt{245}}{5}$

$h = \frac{7\sqrt{5}}{5}$

6.2. Kružnica

* Jednačina kružnice

$(x-p)^2+(y-q)^2=r^2$ $O_1(p,q)$ - centar, r - poluprečnik

* Tačka $A(x_1,y_1)$ leži na kružnici ako je:

$(x_1-p)^2+(y_1-q)^2=r^2$

* Odnos prave i kruga

- Presek prave i kruga je rešenje sistema koji čine jednačina prave i jednačina kružnice. Uvrštavanjem jedne promenljive iz jednačine prave u jednačinu kružnice dobija se kvadratna jednačina:

1) Ako postoje 2 realna rešenja - prava seče krug u 2 tačke $(D>0)$.
2) Ako postoji samo jedno rešenje - prava dodiruje krug $(D=0)$.
3) Ako nema realnih rešenja - prava je van kruga $(D<0)$.

4. I

$A(2,3)$

$B(5,2)$

$O_1(p,0)$

$k_{AB}=\frac{2-3}{5-2}=\frac{-1}{3}$

$k_s=-\frac{1}{k_{AB}}=3$

$S\left(\frac{2+5}{2},\frac{3+2}{2}\right)=\left(\frac{7}{2},\frac{5}{2}\right)$

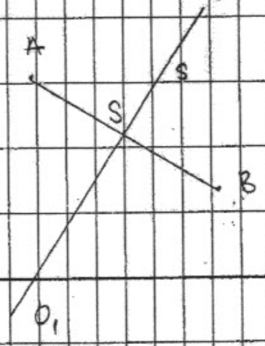
$s:\ y-\frac{5}{2}=3\left(x-\frac{7}{2}\right)$

$0-\frac{5}{2}=3x-\frac{21}{2}$

$3x=\frac{16}{2}=8$

$x=\frac{8}{3}$

$O_1\left(\frac{8}{3},0\right)$



$r=O_1A=\sqrt{\left(2-\frac{8}{3}\right)^2+(3-0)^2}$

$r=\sqrt{\frac{4}{9}+\frac{81}{9}}$

$r=\sqrt{\frac{85}{9}}$

$k:\ \left(x-\frac{8}{3}\right)^2+y^2=\frac{85}{9}$

II $(x-p)^2+(y-0)^2=r^2$

$(2-p)^2+3^2=r^2$

$(5-p)^2+2^2=r^2$

$\cancel{4}-4p+\cancel{p^2}+9=25-10p\cancel{+p^2}\cancel{+4}$

$6p=16$

$p=\frac{\cancel{16}^{8}}{\cancel{6}_{3}}=\frac{8}{3}$

$k:\ \left(x-\frac{8}{3}\right)^2+y^2=\frac{85}{9}$

5. $y = x + 4$

$x^2 + y^2 + 6x - 4y = 0$

$x^2 + 6x + 9 - 9 + y^2 - 4y + 4 - 4 = 0$

$(x+3)^2 + (y-2)^2 = 13$

$O_1(-3, 2)$

$r = \sqrt{13}$

$y = x + 4$

$x^2 + y^2 + 6x - 4y = 0$

$x^2 + x^2 + 8x + \cancel{16} + 6x - 4x - \cancel{16} = 0$

$2x^2 + 10x = 0$

$2x(x+5) = 0$

$2x = 0 \quad \vee \quad x + 5 = 0$

$x_1 = 0 \qquad x_2 = -5$

$y_1 = 4 \qquad y_2 = -1$

$T_1(0, 4) \qquad T_2(-5, -1)$

3 6. $O_1(6, 4)$

$p: 4x - 12y + 144 = 0$

$12y = 4x + 144$

$y = \frac{1}{3}x + 12$

$k_p = \frac{1}{3}$

$k_n = -3$

$n: y - 4 = -3(x - 6)$

$y - 4 = -3x + 18$

$y = -3x + 22$

$y = \frac{1}{3}x + 12$

$y = -3x + 22$

$\frac{1}{3}x + 12 = -3x + 22$

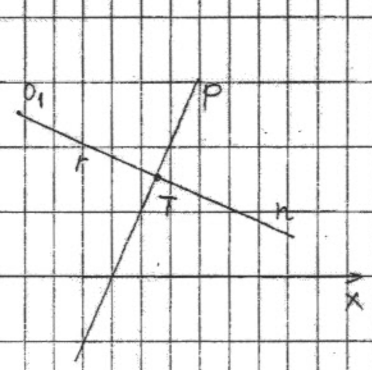
$x + 36 = -9x + 66$

$10x = 30$

$x = 3$

$y = 13$

4 $T(3, 13)$



$r^2 = O_1T^2 = (-3)^2 + 9^2 = 9 + 81 = 90$

$k: (x-6)^2 + (y-4)^2 = 90$

7. $k: x^2 + y^2 - 6x - 8y + 17 = 0$

$p: x + y - 11 = 0$

$x^2 - 6x + 9 - 9 + y^2 - 8y + 16 - 16 + 17 = 0$

$(x-3)^2 + (y-4)^2 = 8$

$r^2 = 8$

$r = \sqrt{8}$

$O_1(3, 4)$

$p: x = 11 - y$

$k: (11-y)^2 + y^2 - 66 + 6y - 8y + 17 = 0$

$121 - 22y + y^2 + y^2 - 66 + 6y - 8y + 17 = 0$

$2y^2 - 24y + 72 = 0 \quad |:2$

$y^2 - 12y + 36 = 0$

$y_{1,2} = \frac{12 \pm \sqrt{144 - 144}}{2}$

$y = 6$

$x = 5$

$D = 0 \Rightarrow p$ je tangenta kružnice k.

c) $x^2+y^2-8x-6y+17=0$

$M(6, y_0),\ y_0>3$

$36+y^2-48-6y+17=0$

$y^2-6y+5=0$

$y_{1,2}=\dfrac{6\pm\sqrt{36-20}}{2}$ $\begin{matrix}\to y_1=5 \Rightarrow M(6,5)\\ \to \cancel{y_2=1}\quad y_0>3\end{matrix}$

I $x^2-8x+16-16+y^2-6y+9-9+17=0$

$(x-4)^2+(y-3)^2=8$

$O_1(4,3)$

$O_1M=m$

$k_m=\dfrac{3-5}{4-6}=1$

$k_t=-1$

$t:\ y-5=-(x-6)$

$y-5=-x+6$

$y=-x+11$

II $y-5=k(x-6)$

$y-5=kx-6k$

$y=kx-6k+5$

$k:\ x^2+(kx-6k+5)^2-8x-6(kx-6k+5)+17=0$

$x^2+k^2x^2+2kx(-6k+5)+(-6k+5)^2-8x-6kx+36k-30+17=0$

$x^2+k^2x^2-12k^2x+10kx+36k^2+2(-6k)\cdot 5+25-8x-6kx+36k-30+17=0$

$x^2+k^2x^2-12k^2x+10kx+36k^2-60k+25-8x-6kx+36k-30+17=0$

$x^2(1+k^2)+x(-12k^2+10k-8-6k)+36k^2-24k+12=0$

$x^2(1+k^2)+x(-12k^2+4k-8)+36k^2-24k+12=0$

$D=0$

$(-12k^2+4k-8)^2-4(1+k^2)(36k^2-24k+12)=0$

$(-4)^2(3k^2-k+2)^2-4\cdot 12(1+k^2)(3k^2-2k+1)=0$

$$16(3k^2-k+2)^2-48(1+k^2)(3k^2-2k+1)=0$$
$$16(9k^4+6k^2(-k+2)+(-k+2)^2-(3+3k^2)(3k^2-2k+1))=0$$
$$16(\cancel{9k^4}-\cancel{6k^3}+12k^2+k^2-4k+4-9k^2+6k-3-\cancel{9k^4}+\cancel{6k^3}-3k^2)=0$$
$$16(k^2+2k+1)=0$$
$$16(k+1)^2=0$$
$$(k+1)^2=0$$
$$k+1=0$$
$$k=-1$$
$$t:\ y=-x+1$$

6.3. Elipsa, parabola i hiperbola

3 * Jednačina elipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ a - velika poluosa, b - mala poluosa

* Jednačina hiperbole: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ a - realna poluosa, b - imaginarna poluosa

grafik ima dve prave, $y=\pm\frac{b}{a}x$ - asimptote

* Jednačina parabole $y^2=2px$ - nastala je rotacijom parabole

4 $y=\frac{1}{2p}x^2$ oko koordinatnog početka za $-90°$.

9. $x+y+\lambda=0$
$2x^2+3y^2=30 \quad |:30$

$\frac{x^2}{15}+\frac{y^2}{10}=1$ - elipsa $y=-x-\lambda$

$a=\sqrt{15}$
$b=\sqrt{10}$

$$2x^2+3(-x-\lambda)^2=30$$
$$2x^2+3(x^2+2\lambda x+\lambda^2)=30$$
$$2x^2+3x^2+6\lambda x+3\lambda^2=30$$
$$5x^2+6\lambda x+3\lambda^2-30=0$$
$$D=0$$

$$(6\lambda)^2-4\cdot5(3\lambda^2-30)=0$$
$$36\lambda^2-20(3\lambda^2-30)=0$$
$$36\lambda^2-60\lambda^2+600=0$$
$$-24\lambda^2+600=0$$
$$24\lambda^2=600$$

$$\lambda^2=25$$
$$\lambda_1=5$$
$$\lambda_2=-5$$

10. $x^2 - 4y^2 = 4$ $\qquad \frac{x^2}{4} - \frac{y^2}{1} = 1$ - hiperbola

$T(1,0)$

$a = 2$

$b = 1$

$t: \quad y - 0 = k(x-1)$

$y = kx - k$

$x^2 - 4(kx-k)^2 - 4 = 0$

$x^2 - 4(k^2x^2 - 2k^2x + k^2) - 4 = 0$

$x^2 - 4k^2x^2 + 8k^2x - 4k^2 - 4 = 0$

$x^2(1-4k^2) + 8k^2x - 4(k^2+1) = 0$

$D = 0$

$(8k^2)^2 - 4(1-4k^2)(-4(k^2+1)) = 0$

$64k^4 + 16(1-4k^2)(k^2+1) = 0$

$64k^4 + (16 - 64k^2)(k^2+1) = 0$

$\cancel{64k^4} + 16k^2 + 16 - \cancel{64k^4} - 64k^2 = 0$

$48k^2 = 16$

$k^2 = \frac{1}{3}$

$k_{1,2} = \pm \frac{\sqrt{3}}{3}$

$t_1: y = \frac{\sqrt{3}}{3}(x-1) \qquad t_2: y = -\frac{\sqrt{3}}{3}(x-1)$

$\operatorname{tg}\varphi = \frac{k_2 - k_1}{1 + k_1 k_2}$

$\operatorname{tg}\varphi = \frac{\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3}}{1 + \frac{\sqrt{3}}{3}\left(-\frac{\sqrt{3}}{3}\right)} = \frac{\frac{2\sqrt{3}}{3}}{1 - \frac{3}{9}} = \frac{\frac{2\sqrt{3}}{\cancel{3}}}{\frac{\cancel{6}}{\cancel{9}}} = \sqrt{3}$

$\varphi = 60°$

$(1-4k^2)x^2+8k^2x-4(k^2+1)=0 \qquad k_{1,2}=\pm\frac{\sqrt{3}}{3} \qquad k^2=\frac{1}{3}$

$(1-4\cdot\frac{1}{3})x^2+8\,\frac{1}{3}\,x-4\left(\frac{1}{3}+1\right)=0$

$-\frac{1}{3}x^2+\frac{8}{3}x-\frac{16}{3}=0 \quad |\cdot 3$

$-x^2+8x-16=0$

$x_{1,2}=\frac{-8\pm\sqrt{64-64}}{-2}$

$x=4$

$y_1=\frac{\sqrt{3}}{3}(4-1)=\sqrt{3} \qquad y_2=-\frac{\sqrt{3}}{3}(4-1)=-\sqrt{3}$

$M_1(4,\sqrt{3}) \qquad M_2(4,-\sqrt{3})$

11. $y^2=16x \qquad y-4=k(x-1)$

3 $M(x_0,4) \qquad y-4=kx-k$

$y=kx-k+4$

$y_0^2=16x_0 \qquad (kx-k+4)^2=16x$

$16=16x_0 \qquad k^2x^2+2kx(-k+4)+(-k+4)^2=16x$

$x_0=1 \qquad k^2x^2-2k^2x+8kx+k^2-8k+16=16x$

$M(1,4) \qquad k^2x^2+(-2k^2+8k-16)x+k^2-8k+16=0$

4 $D=0$

$(-2k^2+8k-16)^2-4k^2(k^2-8k+16)=0$

$\cancel{4k^4}+2(-2k^2)(8k-16)+(8k-16)^2-\cancel{4k^4}+32k^3-64k^2=0$

$-4k^2(8k-16)+(8k-16)^2+32k^3-64k^2=0$

$-\cancel{32k^3}+\cancel{64k^2}+64k^2-256k+256+\cancel{32k^3}-\cancel{64k^2}=0$

$4(16k^2-64k+64)=0$

$16(k^2-4k+4)=0$

$16(k-2)^2=0$

$k=2$

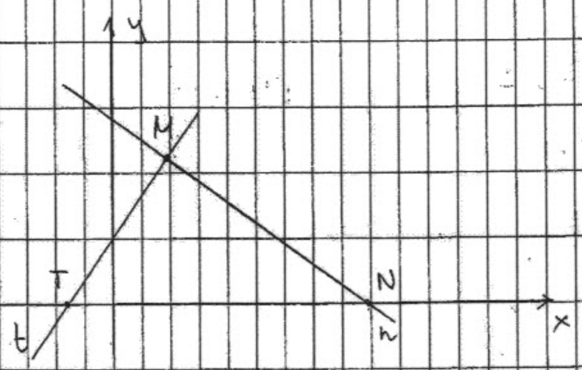
$t: \; y=2x+2$

$k_1 = -\frac{1}{2}$

$y - 4 = -\frac{1}{2}(x-1)$

$y - 4 = -\frac{1}{2}x + \frac{1}{2}$

$y = -\frac{1}{2}x + \frac{9}{2}$



$T(x_1, 0) \in t \Rightarrow 0 = 2x_1 + 2$

$2x_1 = -2$

$x_1 = -1$

$T(-1, 0)$

$N(x_2, 0) \in n \Rightarrow 0 = -\frac{1}{2}x_2 + \frac{9}{2}$

$\frac{1}{2}x_2 = \frac{9}{2}$

$x_2 = 9$

$N(9, 0)$

$NM = \sqrt{(1-9)^2 + (4-0)^2}$

$NM = \sqrt{64+16}$

$NM = \sqrt{80} = 4\sqrt{5}$

$TM = \sqrt{(1+1)^2 + (4-0)^2}$

$TM = \sqrt{4+16}$

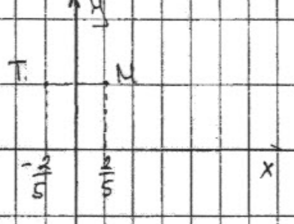
$TM = \sqrt{20} = 2\sqrt{5}$

$P = \frac{NM \cdot TM}{2} = \frac{4\sqrt{5} \cdot 2\sqrt{5}}{\cancel{2}} = 20$

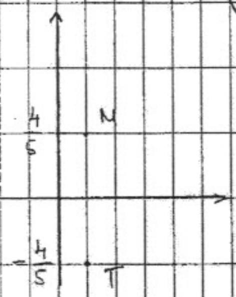
6.4. Zadaci za vežbu

1. $M\left(\frac{2}{5}, \frac{4}{5}\right)$

a) y-osa $\quad T\left(-\frac{2}{5}, \frac{4}{5}\right)$



b) x-osa $\quad T\left(\frac{2}{5}, -\frac{4}{5}\right)$

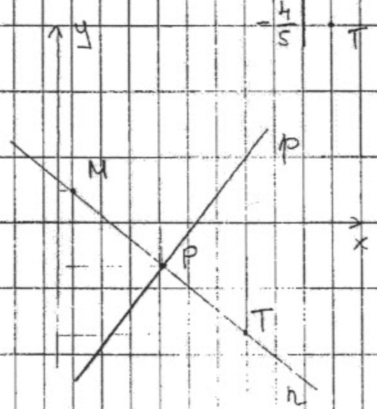


2) $p: x - 3y - 4 = 0$

$3y = x - 4$

$y = \frac{1}{3}x - \frac{4}{3}$

$k = \frac{1}{3}$



$k_n = \frac{-1}{k} = -3$

$n: \ y - \frac{4}{5} = -3\left(x - \frac{2}{5}\right)$

$y - \frac{4}{5} = -3x + \frac{6}{5}$

$y = -3x + 2$

$y = \frac{1}{3}x - \frac{4}{3}$

$y = -3x + 2$

$\frac{1}{3}x - \frac{4}{3} = -3x + 2 \quad / \cdot 3$

$x - 4 = -9x + 6$

$10x = 10$

$x = 1$

$y = -1$

$P(1, -1)$

$\vec{PM} = \vec{TP}$

$\left(\frac{2}{5} - 1, \frac{4}{5} + 1\right) = (1 - x, -1 - y)$

$\left(-\frac{3}{5}, \frac{9}{5}\right) = (1 - x, -1 - y)$

$1 - x = -\frac{3}{5} \qquad -1 - y = \frac{9}{5}$

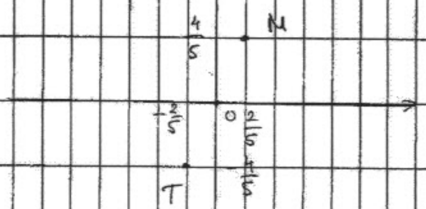
$x = \frac{3}{5} + 1 \qquad y = -1 - \frac{9}{5}$

$x = \frac{8}{5} \qquad y = -\frac{14}{5}$

$T\left(\frac{8}{5}, -\frac{14}{5}\right)$

a) koordinatni početak

3 $T\left(-\frac{2}{5}, -\frac{4}{5}\right)$



2. $T(-1, -1)$

4 $p: 3x + 2y = 6$

$\operatorname{tg} \varphi = \frac{1}{2}$

$p: 2y = -3x + 6$

$y = -\frac{3}{2}x + 3$

$k_1 = -\frac{3}{2}$

$\operatorname{tg} \varphi = \frac{k_2 - k_1}{1 + k_1 k_2}$

$\frac{1}{2} = \frac{k_2 + \frac{3}{2}}{1 + \left(-\frac{3}{2}\right)k_2}$

$2\left(k_2 + \frac{3}{2}\right) = 1 - \frac{3}{2}k_2$

$2k_2 + 3 = 1 - \frac{3}{2}k_2$

$\frac{7}{2}k_2 = -2$

$k_2 = \frac{-\frac{2}{1}}{\frac{7}{2}}$

$k_2 = -\frac{4}{7}$

$y + 1 = -\frac{4}{7}(x + 1)$

$y + 1 = -\frac{4}{7}x - \frac{4}{7}$

$p_1: y = -\frac{4}{7}x - \frac{11}{7}$

$\operatorname{tg} \varphi = \frac{k_1 - k_2}{1 + k_1 k_2}$

$\operatorname{tg} \varphi = \frac{-\frac{3}{2} - k_2}{1 + \left(-\frac{3}{2}\right)k_2}$

$\frac{1}{2} = \frac{-\frac{3}{2} - k_2}{1 + \left(-\frac{3}{2}\right)k_2}$

$-3 - 2k = 1 - \frac{3}{2}k_2$

$-\frac{1}{2}k = 4$

$k = -8$

$y + 1 = -8(x + 1)$

$y + 1 = -8x - 8$

$p_2: y = -8x - 9$

3. $(2a+5)x-(4a-1)y+21=0$

$2ax-(a+1)y+14=0$

$x=0$

$-(4a-1)y=-21$

$-(a+1)y=-14$

$\frac{4a-1}{a+1}=\frac{21}{14}=\frac{3}{2}$

$8a-2=3a+3$

$5a=5$

$a=1$

4. $M(3,4)$ $N(1,2)$

$\vec{TN}=\vec{MN}$

$(3-x,4-y)=(-2,-2)$

$3-x=-2$ $4-y=-2$

$x=5$ $y=6$

$T(5,6)$

$k_1=\frac{2-4}{1-3}$

$k_1=1$

$k_2=-1$

$y-6=-1(x-5)$

$y-6=-x+5$

$y=-x+11$

5. $T(1,1)$

$g: 3x-y+1=0$

$y=3x+1$

$k=3$

$tg\alpha=3$

$tg\alpha=\frac{k_2-k_1}{1+k_1k_2}$

$3=\frac{k_2-3}{1+3k_2}$

$k_2-3=3+9k_2$

$-8k_2=6$

$k_2=\frac{-3}{4}$

$p: y-1=-\frac{3}{4}(x-1)$

$y-1=-\frac{3}{4}x+\frac{3}{4}$

$y=-\frac{3}{4}x+\frac{7}{4}$

$\bar{F}$

$\bar{A}$

$\bar{A}$

$\bar{A}$

$\bar{A}$

6. $A(2,2)$

$y = 2x+1$

$k_p = -\frac{1}{2}$

$p: y-2 = -\frac{1}{2}(x-2)$

$y - 2 = -\frac{1}{2}x + 1$

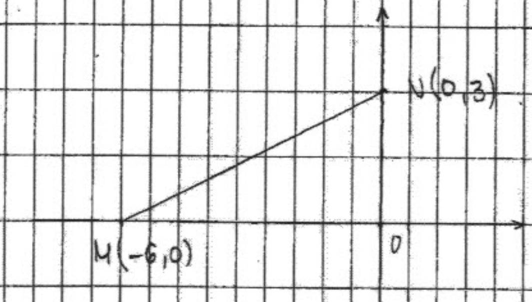
$y = -\frac{1}{2}x + 3$

$y=0 \Rightarrow -\frac{1}{2}x+3=0$

$-\frac{1}{2}x = 3$

$x = -6$

$x = 0 \Rightarrow y = 3$



$|OM| = \sqrt{36} = 6$

$|ON| = \sqrt{9} = 3$

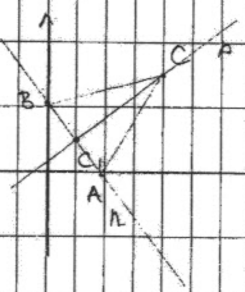
$P_{OMN} = \frac{ON \cdot ON}{2}$

$P_{OMN} = \frac{3 \cdot 6}{2} = 9$

7. $P = 8$

3 $C(5,5)$

$C_1(1,1)$



$p\ \frac{y-1}{5-1} = \frac{x-1}{5-1}$

$4(y-1) = 4(x-1)$

$4y - \cancel{4} = 4x - \cancel{4}$

$4y = 4x$

$y = x$

$k_p = 1$

$k_n = -1$

$n\quad y - 1 = -1(x-1)$

$y - 1 = -x + 1$

$y = -x + 2$

$|CC_1| = \sqrt{16+16}$

$|CC_1| = \sqrt{32} = 4\sqrt{2}$

$\frac{|AB| \cdot |CC_1|}{2} = P$

4 $|AB| = \frac{2P}{|CC_1|} = \frac{16}{4\sqrt{2}} = \frac{4}{\sqrt{2}} = \frac{\cancel{4}^{2}\sqrt{2}}{\cancel{2}} = 2\sqrt{2}$

$\sqrt{2} = \sqrt{(x_1-1)^2 + (y_1-1)^2}$

$y_1 = -x_1 + 2$

$2 = (x_1-1)^2 + (-x_1+2-1)^2$

$\cancel{2} = x_1^2 - 2x_1 + \cancel{1} + x_1^2 - 2x_1 + \cancel{1}$

$2x_1^2 - 4x_1 = 0$

$2x_1(x_1 - 2) = 0$

$x_1 = 0 \lor x_1 = 2$

$\cancel{y_1 = 2} \quad y_1 = 0$

$A(2,0)$

$\sqrt{2} = \sqrt{(x_2-1)^2 + (y_2-1)^2}$

$y_2 = -x_2 + 2$

$2 = (x_2-1)^2 + (y_2-1)^2$

$2x_2(x_2 - 2) = 0$

$x_2 = 0 \lor x_2 = 2$

$y_2 = 2 \quad \cancel{y_2 = 0}$

$B(0,2)$

8. $p: x - 2y + 2 = 0$

$|OA| = 1$

$|OB| = 1$

$2y = x + 2$

$y_{1,2} = \frac{1}{2}x + 1$

$|OA| = \sqrt{x_1^2 + y_1^2}$

$x_1^2 + y_1^2 = 1$

$x_1^2 + \frac{1}{4}x_1^2 + x + \cancel{1} = \cancel{1}$

$\frac{5}{4}x_1^2 + x_1 = 0$

$x_1\left(\frac{5}{4}x_1 + 1\right) = 0$

$x_1 = 0 \quad \vee \quad \cancel{x_1 = -\frac{4}{5}}$

$y_1 = 1 \qquad \cancel{y_1 = \frac{3}{5}}$

$B(0, 1)$

$|OB| = \sqrt{x_2^2 + y_2^2}$

$x_2^2 + y_2^2 = 1$

$x_2\left(\frac{5}{4}x_2 + 1\right) = 0$

$\cancel{x_2 = 0} \quad \vee \quad x_2 = -\frac{4}{5}$

$\cancel{y_2 = 0} \qquad y_2 = \frac{3}{5}$

$A\left(-\frac{4}{5}, \frac{3}{5}\right)$

9. $k \in R$

$p: y = kx + 2$

$k: x^2 - 2x + y^2 + 4y + 1 = 0$

$x^2 - 2x + k^2x^2 + 4kx + 4 + 4kx + 8 + 1 = 0$

$x^2(1 + k^2) + x(-2 + 4k + 4k) + 13 = 0$

$D < 0$

$(8k - 2)^2 - 4(1 + k^2) \cdot 13 = 0$

$64k^2 - 32k + 4 - 52 - 52k^2 = 0$

$12k^2 - 32k - 48 = 0 \quad) : 4$

$3k^2 - 8k - 12 = 0$

$k_{1,2} = \frac{8 \pm \sqrt{64 + 144}}{6}$

$k_1 = \frac{8 + \sqrt{208}}{6} = \frac{8 + 4\sqrt{13}}{6} = \frac{4(2 + \sqrt{13})}{6} = \frac{2(2 + \sqrt{13})}{3}$

$k_2 = \frac{8 - \sqrt{208}}{6} = \frac{8 - 4\sqrt{13}}{6} = \frac{4(2 - \sqrt{13})}{6} = \frac{2(2 - \sqrt{13})}{3}$

$R: x \in \left[\frac{2}{3}(2 - \sqrt{13}), \frac{2}{3}(2 + \sqrt{13})\right]$

10. $P(2,-2)$ $\quad |PQ| = \sqrt{(4-2)^2+(2+2)^2}$

$Q(4,2)$ $\quad |PQ| = \sqrt{20}$

$y=0$ $\quad |PQ| = 2\sqrt{5} = R$

$r = \frac{R}{2} = \sqrt{5}$

$O_1\left(\frac{2+4}{2}, \frac{-2+2}{2}\right)$

$O_1(3,0)$

$k: (x-3)^2+(y-0)^2 = 5 \qquad y=0$

$x^2-6x+9=5$

$x^2-6x+4=0$

$x_{1,2} = \frac{6 \pm \sqrt{36-16}}{2} \to x_1 = \frac{6+2\sqrt{5}}{2} = \frac{\cancel{2}(3+\sqrt{5})}{\cancel{2}}$

3 $\to x_2 = \frac{6-2\sqrt{5}}{2} = \frac{\cancel{2}(3-\sqrt{5})}{\cancel{2}}$

$A(3+\sqrt{5}, 0)$

$B(3-\sqrt{5}, 0)$

11. $x^2+(y+2)^2=4$ $\qquad t: (1+k^2)r^2 = (q-kp-n)^2$

$p: 3x-4y+3=0$ $\qquad \left(1+\frac{16}{9}\right)\cdot 4 = (-2-n)^2$

$4y = 3x+3$ $\qquad \frac{100}{9} = (n+2)^2$

$y = \frac{3}{4}x + \frac{3}{4}$

4 $k_p = \frac{3}{4}$ $\qquad \frac{10}{3} = n_1+2 \qquad n_2+2 = -\frac{10}{3}$

$k_t = -\frac{4}{3}$ $\qquad n_1 = \frac{4}{3} \qquad n_2 = -\frac{16}{3}$

$t_1: y = -\frac{4}{3}x + \frac{4}{3} \qquad t_2: y = -\frac{4}{3}x - \frac{16}{3}$

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12. $y = -\frac{x}{2}$ $\qquad 4x = \left(-\frac{x}{2}\right)^2$

$y^2 = 4x$ $\qquad 4x = \frac{x^2}{4} \;\Big|\cdot 4$ $\qquad A(0,0)$

$x^2-16x=0$ $\qquad B(16,-8)$

$x(x-16)=0$ $\qquad S\left(\frac{0+16}{2}, \frac{0-8}{2}\right)$

$x=0 \qquad x-16=0$ $\qquad S(8,-4)$

$y=0 \qquad x=16$

$y=-8$

13. $y+x+n=0$

$2x^2+3y^2=30$

$y=-x-n$

$2x^2+3(-x-n)^2=30$

$2x^2+3(x^2+2xn+n^2)-30=0$

$2x^2+3x^2+6xn+3n^2-30=0$

$x^2(2+3)+x6n+3n^2-30=0$

$D=0$

$36n^2-4\cdot5\cdot(3n^2-30)=0$

$36n^2-60n^2+600=0$

$-24n^2+600=0$

$n^2=\frac{600}{24}=25$

$n_1=5 \quad n_2=-5$

$n\in\{-5,5\}$

14. $y=kx+2$

$x^2-2y^2=1$

$x^2-2(kx+2)^2=1$

$x^2-2(k^2x^2+4kx+4)-1=0$

$x^2-2k^2x^2-8kx-8-1=0$

$x^2(1-2k^2)-8kx-9=0$

$D>0$

$64k^2-4(1-2k^2)\cdot(-9)=0$

$64k^2+36(1-2k^2)=0$

$64k^2+36-72k^2=0$

$-8k^2+36=0$

$8k^2=36$

$k^2=\frac{36}{8}$

$k_1=+\frac{6}{2\sqrt{2}}=\frac{3\sqrt{2}}{2} \qquad k_2=-\frac{3\sqrt{2}}{2}$

$k\in\left(-\frac{3\sqrt{2}}{2},\frac{3\sqrt{2}}{2}\right)$

15. $k: x^2+y^2-24=0$

$p: 7x-y=0$

$\varphi=45°$

$y=7x$

$k=7$

$tg\varphi=\frac{k_1-k}{1+kk_1}$

$1=\frac{k_1-7}{1+7k_1}$

$k_1-7=1+7k_1$

$-6k_1=8$

$k_1=-\frac{4}{3}$

$tg\varphi=\frac{k-k_2}{1+k_1k_2}$

$1=\frac{7-k_2}{1+k_27}$

$7-k_2=1+7k_2$

$-8k_2=-6$

$k_2=\frac{3}{4}$

$t_{1,2}: (1+k_1^2)r^2=(q-kp-n)^2$

$\left(1+\frac{16}{9}\right)24=(-n)^2$

$\frac{600}{9}=(-n)^2$

$-\frac{10\sqrt{6}}{3}=n \qquad \frac{10\sqrt{6}}{3}=n$

3 $t_1: y=-\frac{4}{3}x-\frac{10\sqrt{6}}{3}$

$t_2: y=-\frac{4}{3}x+\frac{10\sqrt{6}}{3}$

$t_{3,4}: \left(1+\frac{9}{16}\right)24=(-n)^2$

$\frac{600}{16}=(-n)^2$

$n=\frac{10\sqrt{6}}{4}=\frac{5\sqrt{6}}{2} \qquad n=-\frac{5\sqrt{6}}{2}$

$t_3: y=\frac{3}{4}x+\frac{5\sqrt{6}}{2}$

$t_4: y=\frac{3}{4}x-\frac{5\sqrt{6}}{2}$

16. $x^2-4y^2=16$

$T(5, y_0)$

$y_0>0$

4

$25-4y^2=16$

$4y^2=9$

$y^2=\frac{9}{4}$

$y=\frac{3}{2} \ \vee \ \cancel{y=-\frac{3}{2}} \Rightarrow y>0$

$T\left(5, \frac{3}{2}\right)$

$t: y-\frac{3}{2}=k(x-5)$

$y-\frac{3}{2}=kx-5k$

$y=kx-5k+\frac{3}{2}$

$x^2-4\left(kx-5k+\frac{3}{2}\right)^2=16$

$x^2-4\left(k^2x^2+2kx\left(-5k+\frac{3}{2}\right)+\left(-5k+\frac{3}{2}\right)^2\right)-16=0$

$x^2-4\left(k^2x^2-10k^2x+3kx+25k^2-15k+\frac{9}{4}\right)-16=0$

$x^2-4k^2x^2+40k^2x-12kx-100k^2+60k-9-16=0$

$x^2(1-4k^2)+x(40k^2-12k)-100k^2+60k-25=0$

$D=0$

$(40k^2-12k)^2-4(1-4k^2)(-100k^2+60k-25)=0$

$4^2(10k^2-3k)^2-(4-16k^2)(-100k^2+60k-25)=0$

$16(100k^4-60k^3+9k^2)+400k^2-240k+100-1600k^4+960k^3-400k^2=0$

$\cancel{1600k^4}-\cancel{960k^3}+144k^2+\cancel{400k^2}-240k+100-\cancel{1600k^4}+\cancel{960k^3}-\cancel{400k^2}=0$

$144k^2-240k+100=0 \quad |:4$

$36k^2-60k+25=0$

$k_{1,2}=\frac{60\pm\sqrt{3600-3600}}{72}$

$k=\frac{60}{72}=\frac{5}{6}$

$t: y=\frac{5}{6}x-\frac{25}{6}+\frac{3}{2}$

$y=\frac{5}{6}x-\frac{16}{6}$

$y=\frac{5}{6}x-\frac{8}{3}$

17. $x^2+3y^2=12$

$T(0,4)$

$t: y-4=kx$

$y=kx+4$

$x^2+3(kx+4)^2=12$

$x^2+3(k^2x^2+8kx+16)-12=0$

$x^2+3k^2x^2+24kx+48-12=0$

$x^2(1+3k^2)+x\,24k+36=0$

$D=0$

$576k^2-144(1+3k^2)=0$

$576k^2-144-432k^2=0$

$144k^2=144$

$k^2=1$

$k_1=1 \quad k_2=-1$

$tg\varphi=\frac{k_2-k_1}{1+k_1k_2}$

$tg\varphi=\frac{-2}{0}=\infty$

$\varphi=90^\circ=\frac{\pi}{2}$

18. $y^2 = 16x$

$\varphi = 135°$

$tg\varphi = tg\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$

$tg\varphi = -ctg\left(\frac{\pi}{4}\right)$

$tg\varphi = -1$

$k = tg\varphi = -1$

$y = -x + n$

uslov za tangentu: $2kn = p$

$-2n = 8$

$n = -4$

$y = -x - 4$

$(-x-4)^2 = 16x$

$x^2 + 8x + 16 - 16x = 0$

$x^2 - 8x + 16 = 0$

$(x-4)^2 = 0$

$x = 4$

3 $y = -8$

$T(4, -8)$

19. $(x-2)^2 + y^2 = 1$

$A(0, 0)$

$t: y - 0 = k(x - 0)$

$y = kx$

$(x-2)^2 + k^2x^2 - 1 = 0$

$k^2x^2 + x^2 - 4x + 4 - 1 = 0$

$x^2(k^2+1) - 4x + 3 = 0$

$D = 0$

4 $16 - 4 \cdot 3(k^2+1) = 0$

$16 - 12k^2 - 12 = 0$

$12k^2 = 4$

$k^2 = \frac{4}{12}$

$k_1 = \frac{\cancel{2}}{\cancel{2}\sqrt{3}} = \frac{1}{\sqrt{3}}$ $\qquad k_2 = -\frac{1}{\sqrt{3}}$

$t_1: y = \frac{x}{\sqrt{3}}$ $\qquad t_2: y = -\frac{x}{\sqrt{3}}$

20. $p: 2x-3y+4=0$

$y^2=4x$

$3y=2x+4$

$y=\frac{2}{3}x+\frac{4}{3}$

$2x=3y-4$

$x=\frac{3}{2}y-2$

$y^2=4\cdot\left(\frac{3}{2}y-2\right)$

$y^2-6y+8=0$

$y_{1,2}=\frac{6\pm\sqrt{36-32}}{2}$

$y_1=4 \qquad y_2=2$

$x_1=4 \qquad x_2=1$

tangente: $y_0 y=p(x+x_0)$

$t_1: 4y=2(x+4)$

$4y=2x+8$

$y=\frac{1}{2}x+2$

$t_2: 2y=2(x+1)$

$2y=2x+2$

$y=x+1$

21. $A(1,2)$

$B(2,3)$

$C(2,5)$

$p: \frac{y-2}{3-2}=\frac{x-1}{2-1}$

$y-2=x-1$

$y=x+1$

$k=1$

$k_n=-1$

$O_1=C(2,5)$

$n: y-5=-(x-2)$

$y-5=-x+2$

$y=-x+7$

$y=x+1$

$y=-x+7$

$x+1=-x+7$

$2x=6$

$x=3$

$y=4$

$D(3,4)$

$|CD|=r$

$|CD|=\sqrt{(3-2)^2+(4-5)^2}$

$r=\sqrt{2}$

$k: (x-2)^2+(y-5)^2=2$

23. $A(1,1)$

$B(2,1)$

$C(2,1+\sqrt{3})$

$k: (x-1)^2+y^2=1$

$k_1=\frac{1-1}{2-1}=0$

$k_2=\frac{1+\sqrt{3}-1}{2-1}$

$k_2=\sqrt{3}$

$tg\,\varphi=\frac{k_2-k_1}{1+k_1k_2}=\sqrt{3}$

$\varphi=60°=\frac{\pi}{3}$

$\varphi_1=\frac{\varphi}{2}=\frac{\pi}{6}$

$k=tg\,\varphi_1=\frac{\sqrt{3}}{3}$

$t: (1+k^2)r^2=(q-kp-u)^2$

$\left(1+\frac{3}{9}\right)\cdot 1=\left(0-\frac{\sqrt{3}}{3}-u\right)^2$

$\frac{4}{3}=\left(-\frac{\sqrt{3}}{3}-u\right)^2$

$-\frac{\sqrt{3}}{3}-u=\frac{2}{\sqrt{3}} \quad \vee \quad -\frac{\sqrt{3}}{3}-u=-\frac{2}{\sqrt{3}}$

$u=-\frac{\sqrt{3}}{3}-\frac{2\sqrt{3}}{3} \qquad u=-\frac{\sqrt{3}}{3}+\frac{2\sqrt{3}}{3}$

$u=\frac{-3\sqrt{3}}{3} \qquad u=\frac{\sqrt{3}}{3}$

$u=-\sqrt{3}$

$t_1: y=\frac{\sqrt{3}}{3}x-\sqrt{3}$

$t_2: y=\frac{\sqrt{3}}{3}x+\frac{\sqrt{3}}{3}$