5 Vektorski račun

5.1. Slobodni vektori

* Vektor (slobodni vektor) je skup svih orijentisanih duži koje su međusobno paralelne, podudarne i isto orijentisane.
- Dužina te duži je intenzitet tog vektora, pravac te duži je pravac tog vektora i smer te duži je smer tog vektora.
- orijentisana duž AB = vektor $\overrightarrow{AB}$
- tačka = nula-vektor.
- Nula-vektor ima isti početak i kraj, $A \equiv B \Rightarrow \overrightarrow{AB} = \vec{0}$
- Vektor: - pravac
  - smer
  - intenzitet (dužina)
- Pravac je skup svih tačaka koje su međusobno paralelne.

$\vec{a} = \vec{b}$ samo ako oba vektora imaju isti pravac, smer i intenzitet.

* Množenje vektora skalarom
- Množenjem vektora $\vec{a}$ realnim brojem α dobija se vektor $\vec{b}$ istog pravca, istog smera za $\alpha > 0$, a suprotnog za $\alpha < 0$, a intenzitet $|\vec{b}|$ vektora $\vec{b}$ je jednak proizvodu broja $|\alpha|$ i intenziteta $|\vec{a}|$ vektora $\vec{a}$.
- Nula puta bilo koji vektor je nula-vektor i bilo koji broj puta nula-vektor je nula-vektor, tj. $0 \cdot \vec{a} = 0$ i $\alpha \cdot \vec{0} = \vec{0}$

* Kolinearnost vektora

Vektori su istog pravca $\Leftrightarrow$ Vektor su kolinearni $\Leftrightarrow$ Vektori leže na istoj pravoj $\Leftrightarrow$ Vektori su paralelni.
- Nula vektor je kolinearan sa svakim vektorom i normalan na svaki vektor.
- Kolinearnost vektora $\vec{a}$ i $\vec{b}$ označava se sa $\vec{a} \parallel \vec{b}$

Ako su $\vec{a}$ i $\vec{b}$ kolinearni vektori, onda se bar jedan od njih može zapisati kao proizvod nekog skalara i onog drugog vektora.

* Sabiranje i oduzimanje

Zbir vektora $\vec{a}$ i $\vec{b}$ dobijamo tako što uzimamo da se vrh vektora $\vec{a}$ poklapa sa početkom vektora $\vec{b}$ i tada je njihov zbir vektor čija se početna tačka poklapa sa početnom tačkom vektora $\vec{a}$, a vrh poklapa se vrhom vektora $\vec{b}$.

Vektori se mogu sabirati i po principu paralelograma.

Oduzimanje vektora se svodi na sabiranje na sledeći način:

$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

* Komplanarnost vektora

Pravci vektora su paralelni jednoj ravni $\Leftrightarrow$ Vektori su komplanarni $\Leftrightarrow$ Vektori leže u istoj ravni $\Leftrightarrow$ Vektori leže u paralelnim ravnima.

- Nula-vektor je komplanaran sa svakim skupom komplanarnih vektora

* Za sve realne brojeve α i β i sve vektore $\vec{a}, \vec{b}$ i $\vec{c}$ važi:

a) $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$

b) $\vec{a}+0 = 0+\vec{a} = \vec{a}$

c) $\vec{a}+(-\vec{a}) = -\vec{a}+\vec{a} = 0$

d) $\alpha(\beta\vec{a}) = (\alpha\beta)\vec{a}$

e) $\alpha(\vec{a}+\vec{b}) = \alpha\vec{a}+\alpha\vec{b}$

f) $(\alpha+\beta)\vec{a} = \alpha\vec{a}+\beta\vec{a}$

g) $1 \cdot \vec{a} = \vec{a}$

* Za svaka dva nekolinearna vektora $\vec{a}$ i $\vec{b}$ važi:

a) $(\forall\alpha,\beta\in R)(\alpha\vec{a}+\beta\vec{b} = 0 \Rightarrow \alpha = \beta = 0)$

b) Za svaki vektor $\vec{c}$ koji je komplanaran sa $\vec{a}$ i $\vec{b}$ je $\vec{c} = \alpha\vec{a}+\beta\vec{b}$ za neke $\alpha, \beta \in R$, pri čemu su α i β jednoznačno određeni.

* Vektori $\vec{a}, \vec{b}$ i $\alpha\vec{a} + \beta\vec{b}$ su komplanarni za svako $\alpha, \beta \in R$.

* Za svaka tri nekomplanarna vektora $\vec{a}, \vec{b}, \vec{c}$ važi:

a) $\forall \alpha, \beta, \gamma \in R \ (\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = 0 \Rightarrow \alpha = \beta = \gamma = 0)$

b) Za svaki vektor $\vec{d}$ je $\vec{d} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$ pri čemu su α, β, γ jednoznačno određeni.

1. Neka je ABCDEF pravilni šestougao, P i Q središte redom stranica BC i EF i tačka T presek duži AP i BQ. U kom odnosu tačka T deli duži AP i BQ?

$\vec{AB} + \vec{BT} + \vec{TA} = 0 \qquad \vec{AB} = \vec{a} \quad \vec{BC} = \vec{b}$

$\vec{PA} = \vec{PB} + \vec{BA} = -\frac{1}{2}\vec{b} - \vec{a}$

$\vec{AF} = \vec{b} - \vec{a}$

$\vec{BQ} = \vec{BA} + \vec{AF} + \vec{FQ} = -\vec{a} + \vec{b} - \vec{a} + \frac{1}{2}\vec{b} = -2\vec{a} + \frac{3}{2}\vec{b}$

$\vec{TA} = \alpha\vec{PA}$

$\vec{BT} = \beta\vec{BQ}$

$\vec{TA} = \alpha\left(-\frac{1}{2}\vec{b} - \vec{a}\right)$

$\vec{BT} = \beta\left(-2\vec{a} + \frac{3}{2}\vec{b}\right)$

$\vec{AB} + \vec{BT} + \vec{TA} = 0$

$\vec{a} + \beta\left(-2\vec{a} + \frac{3}{2}\vec{b}\right) + \alpha\left(-\frac{1}{2}\vec{b} - \vec{a}\right) = 0$

$\vec{a} - 2\beta\vec{a} + \frac{3}{2}\beta\vec{b} - \frac{1}{2}\alpha\vec{b} - \alpha\vec{a} = 0$

$\vec{a}(1 - 2\beta - \alpha) + \vec{b}\left(\frac{3}{2}\beta - \frac{1}{2}\alpha\right) = 0$

$1 - 2\beta - \alpha = 0$

$\frac{3}{2}\beta - \frac{1}{2}\alpha = 0 \ / \cdot 2$

$2\beta + \alpha = 1$

$3\beta - \alpha = 0$

$5\beta = 1$

$\beta = \frac{1}{5} \qquad BT : TQ = 1 : 4$

$\alpha = \frac{3}{5} \qquad AT : TP = 3 : 2$

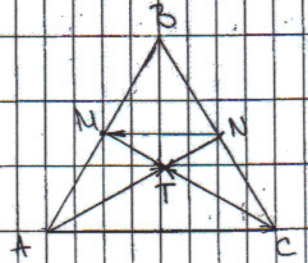
2. Ako su M i N središte stranica AB i BC trougla ABC, pokazati da važi jednakost $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AC}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$, $\overrightarrow{MB} = \frac{1}{2}\overrightarrow{AB}$, $\overrightarrow{BN} = \frac{1}{2}\overrightarrow{BC}$

$\overrightarrow{MN} = \overrightarrow{MB} + \overrightarrow{BN} = \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}) = \frac{1}{2}\overrightarrow{AC}$

$MN \parallel AC$ i $MN = \frac{1}{2}AC$

3. Pokazati da težište deli težišnu liniju u odnosu 2 : 1.



$\overrightarrow{AC} = \overrightarrow{AT} + \overrightarrow{TC}$

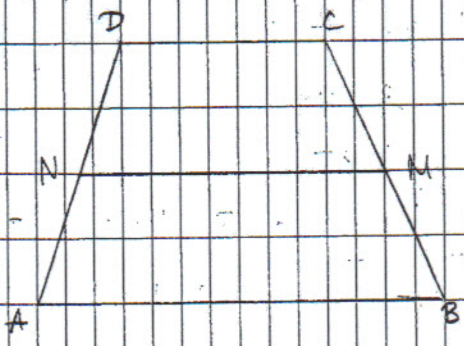
$\overrightarrow{NM} = \overrightarrow{NT} + \overrightarrow{TM}$

$\overrightarrow{NM} = -\overrightarrow{MN} = -\frac{1}{2}\overrightarrow{AC} = -\frac{1}{2}\overrightarrow{AT} - \frac{1}{2}\overrightarrow{TC} = \frac{1}{2}\overrightarrow{TA} + \frac{1}{2}\overrightarrow{CT}$

$\overrightarrow{NT} = \frac{1}{2}\overrightarrow{TA}$ i $\overrightarrow{TM} = \frac{1}{2}\overrightarrow{CT}$

$AT = \frac{2}{3}AN$, $TN = \frac{1}{3}AN$, AN = težišna linija, T = težište trougla

4. Ako su M i N središte stranica BC i AD četvorougla ABCD onda je $2\overrightarrow{MN} = \overrightarrow{CD} + \overrightarrow{BA}$.



$\overrightarrow{MN} = \overrightarrow{MC} + \overrightarrow{CD} + \overrightarrow{DN}$

$\overrightarrow{MN} = \overrightarrow{MB} + \overrightarrow{BA} + \overrightarrow{AN}$

$2\overrightarrow{MN} = \overrightarrow{MC} + \overrightarrow{CD} + \overrightarrow{DN} + \overrightarrow{MB} + \overrightarrow{BA} + \overrightarrow{AN}$ $\quad \overrightarrow{MC} = -\overrightarrow{MB}$

$\overrightarrow{DN} = -\overrightarrow{AN}$

$2\overrightarrow{MN} = \overrightarrow{CD} + \overrightarrow{BA}$

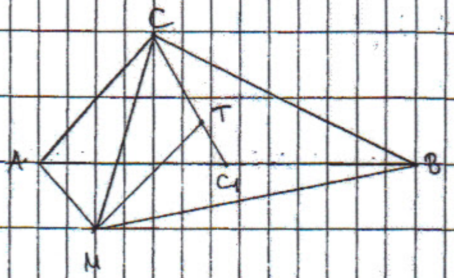
$MN = s \quad AB = a \quad CD = b$

$2s = a + b$

$s = \frac{a+b}{2}$

5. Neka je M proizvoljna tačka, a T težište trougla ABC. Pokazati da je

$\vec{MT} = \frac{1}{3}(\vec{MA} + \vec{MB} + \vec{MC})$



$\vec{MT} = \vec{MA} + \vec{AC} + \vec{CT}$

$\vec{MT} = \vec{MC} + \vec{CT}$

$\vec{MT} = \vec{MB} + \vec{BC} + \vec{CT}$

$3\vec{MT} = \vec{MA} + \vec{MB} + \vec{MC} + \vec{AC} + \vec{BC} + 3\vec{CT}$

$\vec{AC} = \vec{AC_1} + \vec{C_1C}, \ \vec{BC} = \vec{BC_1} + \vec{C_1C}, \ \vec{CT} = \frac{2}{3}\vec{CC_1}$

$3\vec{MT} = \vec{MA} + \vec{MB} + \vec{MC} + \vec{AC_1} + \vec{C_1C} + \vec{BC_1} + \vec{C_1C} + 3 \cdot \frac{2}{3}\vec{CC_1}$

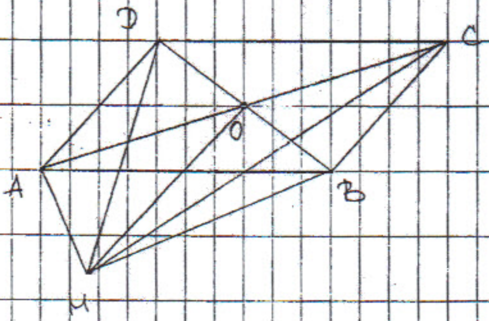
$\vec{AC_1} = -\vec{BC_1} = \frac{1}{2}\vec{AB}, \ \vec{C_1C} = -\vec{CC_1}$

$3\vec{MT} = \vec{MA} + \vec{MB} + \vec{MC} - \vec{BC_1} + \vec{BC_1} + 2\vec{C_1C} - 2\vec{C_1C}$

$3\vec{MT} = \vec{MA} + \vec{MB} + \vec{MC}$

$\vec{MT} = \frac{1}{3}(\vec{MA} + \vec{MB} + \vec{MC})$

6. O je presek dijagonala paralelograma, a M proizvoljna tačka. Dokazati da je $4\vec{MO} = \vec{MA} + \vec{MB} + \vec{MC} + \vec{MD}$



$\vec{MO} = \vec{MA} + \vec{AO}$ $\quad \vec{AO} = \frac{1}{2}\vec{AC}$ $\quad \vec{BO} = \frac{1}{2}\vec{BD}$

$\vec{MO} = \vec{MB} + \vec{BO}$ $\quad \vec{CO} = \frac{1}{2}\vec{CA}$ $\quad \vec{DO} = \frac{1}{2}\vec{DB}$

$\vec{MO} = \vec{MC} + \vec{CO}$ $\quad \vec{AO} = -\vec{CO}$ $\quad \vec{BO} = -\vec{DO}$

$\vec{MO} = \vec{MD} + \vec{DO}$

$4\vec{MO} = \vec{MA} + \vec{MB} + \vec{MC} + \vec{MD} - \vec{CO} - \vec{DO} + \vec{CO} + \vec{DO}$

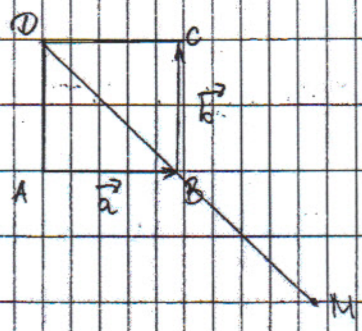
$4\vec{MO} = \vec{MA} + \vec{MB} + \vec{MC} + \vec{MD}$

7. Kod kvadrata ABCD je $\vec{AB} = \vec{a}$ i $\vec{BC} = \vec{b}$. Neka je M tačka takva da je B središte duži DM. Pomoću $\vec{a}$ i $\vec{b}$ izraziti vektore: $\vec{CD}$, $\vec{DA}$, $\vec{AC}$, $\vec{BD}$, $\vec{MA}$, $\vec{MC}$ i $\vec{MD}$.

$\vec{CD} = \vec{BA} = -\vec{AB} = -\vec{a}$

$\vec{DA} = \vec{CB} = -\vec{BC} = -\vec{b}$

$\vec{AC} = \vec{AB} + \vec{BC} = \vec{a} + \vec{b}$



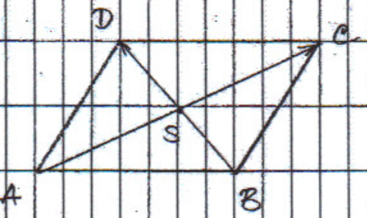
$\vec{BD}=\vec{BC}+\vec{CD}=\vec{b}+(-\vec{a})=\vec{b}-\vec{a}$

$\vec{MA}=\vec{MB}+\vec{BA}=\vec{BD}-\vec{AB}=\vec{b}-\vec{a}-\vec{a}=\vec{b}-2\vec{a}$

$\vec{MC}=\vec{MB}+\vec{BC}=\vec{BD}+\vec{b}=\vec{b}-\vec{a}+\vec{b}=2\vec{b}-\vec{a}$

$\vec{MD}=2\vec{BD}=2\vec{b}-2\vec{a}$

8. Presek dijagonala romba ABCD je S. Izraziti $2\vec{AB}-\frac{1}{4}\vec{AD}$ preko vektora $\vec{u}=\vec{BD}$ i $\vec{v}=\vec{SC}$.



$\vec{AB}=\vec{DC}=\vec{DS}+\vec{SC}=\frac{1}{2}\vec{DB}+\vec{v}=\vec{v}-\frac{1}{2}\vec{u}$

$\vec{AD}=\vec{BC}=\vec{BS}+\vec{SC}=\vec{v}+\frac{1}{2}\vec{u}$

$2\vec{AB}-\frac{1}{4}\vec{AB}=2\left(\vec{v}-\frac{1}{2}\vec{u}\right)-\frac{1}{4}\left(\vec{v}+\frac{1}{2}\vec{u}\right)=$

$=2\vec{v}-\vec{u}-\frac{1}{4}\vec{v}-\frac{1}{8}\vec{u}=$

$=\frac{7}{4}\vec{v}-\frac{9}{8}\vec{u}$

5.2. Vektori u ravni i prostoru

* Ako imamo tačku $A(x,y,z)$, tada je njen vektor položaja $\vec{r}_A=x\vec{i}+y\vec{j}+z\vec{k}\rightarrow$ skraćeno $\vec{r}_A=(x,y,z)$.

* Intenzitet vektora $\vec{r}=(x,y,z)$ je $|\vec{r}|=\sqrt{x^2+y^2+z^2}$

* Sabiranje $(x_1,y_1,z_1)+(x_2,y_2,z_2)=(x_1+x_2,y_1+y_2,z_1+z_2)$

* Množenje vektora skalarom $\lambda(x,y,z)=(\lambda x,\lambda y,\lambda z)$

* Rastojanje između tačaka A i B.

Neka su $A(x_1,y_1,z_1)$ i $B(x_2,y_2,z_2)$ proizvoljne tačke iz prostora. Tada je

$\vec{AB}=\vec{r}_B-\vec{r}_A=(x_2-x_1,y_2-y_1,z_2-z_1)$

$|\vec{AB}|=AB=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

* Deoba duži u datoj razmeri: Ako tačka M deli duž AB u razmeri $\lambda:1$, tj. $\vec{AM}=\lambda\vec{MB}$, tada je $\vec{r}_M=\frac{\vec{r}_A+\lambda\vec{r}_B}{1+\lambda}$

* Na isti način se računa i sa vektorima u ravni, pri čemu njihove

Koordinate predstavljaju uređen par brojeva $\vec{a}=(x_1,y_1)=x_1\vec{i}+y_1\vec{j}$.

9. Za vektore $\vec{a}$ i $\vec{b}$ odrediti $|\vec{a}|$ i $3\vec{a}+2\vec{b}$, gde je:

a) $\vec{a}=3\vec{i}+2\vec{j}-\vec{k}$ $\qquad |\vec{a}|=\sqrt{3^2+2^2+(-1)^2}$

$\vec{b}=-\vec{i}+3\vec{j}-2\vec{k}$ $\qquad |\vec{a}|=\sqrt{9+4+1}=\sqrt{14}$

$3\vec{a}+2\vec{b}=3(3\vec{i}+2\vec{j}-\vec{k})+2(-\vec{i}+3\vec{j}-2\vec{k})=9\vec{i}+6\vec{j}-3\vec{k}-2\vec{i}+6\vec{j}-4\vec{k}=$
$=7\vec{i}+12\vec{j}-7\vec{k}$

b) $\vec{a}=(3,0,-4)$ $\qquad |\vec{a}|=\sqrt{3^2+0+(-4)^2}$

$\vec{b}=(1,0,2)$ $\qquad |\vec{a}|=\sqrt{9+16}=\sqrt{25}=5$

$3\vec{a}+2\vec{b}=3(3,0,-4)+2(1,0,2)=(9,0,-12)+(2,0,4)=(11,0,-8)$

10. a) $\vec{a}=(-2,1,3)$ $\qquad |\vec{a}|=\sqrt{(-2)^2+1^2+3^2}=\sqrt{4+1+9}=\sqrt{14}$

$\vec{b}=(p+1,-p,-3)$ $\qquad |\vec{b}|=\sqrt{(p+1)^2+(-p^2)+(-3)^2}=$

$=\sqrt{p^2+2p+1+p^2+9}$

$=\sqrt{2p^2+2p+10}$

$\sqrt{2p^2+2p+10}=\sqrt{14}$

$2p^2+2p+10=14$

$2p^2+2p-4=0$

$p^2+p-2=0$

$p_{1,2}=\frac{-1\pm\sqrt{1+8}}{2}$ $\to p_1=1$, $p_2=-2$ > isti intenzitet $|\vec{a}|$ i $|\vec{b}|$

$(p+1,-p,-3)=(-2m,m,3m)$ $\qquad \vec{b}=m\vec{a}$

$p+1=-2m \wedge -p=m$

$1-m=-2m$

$m=-1$

$p=1 \longrightarrow$ kolinearni vektori

b) $\vec{a} = 2\vec{i} + p\vec{j} + \vec{k} = (2, p, 1)$

$\vec{b} = 2p\vec{i} + 4\vec{j} - 2\vec{k} = (2p, 4, -2)$

$|\vec{a}| = \sqrt{4 + p^2 + 1} = \sqrt{p^2+5}$

$|\vec{b}| = \sqrt{4p^2 + 16 + 4} = \sqrt{4p^2+20}$

$\sqrt{4p^2+20} = \sqrt{p^2+5}$

$4p^2 + 20 = p^2 + 5$

$3p^2 = -15$

$p^2 = -5$ — Nema rešenja za isti intenzitet

$\vec{b} = m\vec{a}$

$(2p, 4, -2) = (2m, mp, m)$

$2p = 2m$

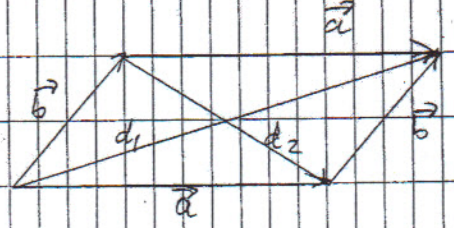
$4 = mp$

$m = -2$

$p = -2 \rightarrow$ kolinearni vektori

11. $\vec{a} = (-1, 1, 0)$

$\vec{b} = (1, -2, -2)$



$\vec{d_1} = \vec{a} + \vec{b} = (-1, 1, 0) + (1, -2, -2)$

$= (0, -1, -2)$

$|\vec{d_1}| = \sqrt{0+1+4} = \sqrt{5}$

$\vec{b} + \vec{d_2} = \vec{a}$

$\vec{d_2} = \vec{a} - \vec{b} = (-1, 1, 0) - (1, -2, -2) = (-2, 3, -2)$

$|\vec{d_2}| = \sqrt{4+9+4} = \sqrt{17}$

12. a) $A(-3,-2,0)$ $\quad \overrightarrow{AB} = (3+3, -3+2, 1-0)$ $\quad \overrightarrow{AC} = (5+3, 0+2, 2-0)$

$B(3,-3,1)$ $\quad \overrightarrow{AB} = (6,-1,1)$ $\quad \overrightarrow{AC} = (8,2,2)$

$C(5,0,2)$ $\quad \overrightarrow{BC} = (5-3, 0+3, 2-1)$

$\overrightarrow{BC} = (2,3,1)$

$c = |\overrightarrow{AB}| = \sqrt{36+1+1} = \sqrt{38}$

$a = |\overrightarrow{BC}| = \sqrt{4+9+1} = \sqrt{15}$

$b = |\overrightarrow{AC}| = \sqrt{64+4+4} = \sqrt{72}$ → Najduža stranica

Njena središnja je točka D:

$$D = \left(\frac{-3+5}{2}, \frac{-2+0}{2}, \frac{0+2}{2}\right) = (1,-1,1)$$

$\overrightarrow{BD} = (-2,2,0)$

$t_b = |\overrightarrow{BD}| = \sqrt{4+4+0} = \sqrt{8} = 2\sqrt{2}$

b) $A(0,0,1)$ $\quad \overrightarrow{AB} = (4,0,0)$ $\quad c = |\overrightarrow{AB}| = \sqrt{16} = 4$ → Najduža stranica

$B(4,0,1)$ $\quad \overrightarrow{BC} = (-2,2,0)$ $\quad a = |\overrightarrow{BC}| = \sqrt{8} = 2\sqrt{2}$

$C(2,2,1)$ $\quad \overrightarrow{AC} = (2,2,0)$ $\quad b = |\overrightarrow{AC}| = \sqrt{8} = 2\sqrt{2}$

$$D = \left(\frac{0+4}{2}, \frac{0+0}{2}, \frac{1+1}{2}\right) = (2,0,1)$$

$\overrightarrow{CD} = (0,-2,0)$

$|\overrightarrow{CD}| = \sqrt{4} = 2$

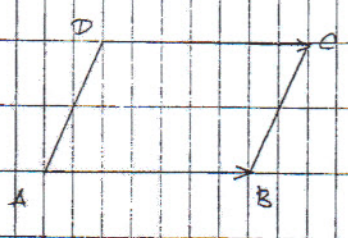
13. $A(1,0,1)$ $\quad \overrightarrow{DC} = \overrightarrow{AB}$

$B(3,1,1)$ $\quad (4-x, 2-y, 3-z) = (2,1,0)$

$C(4,2,3)$ $\quad 4-x=2 \quad 2-y=1 \quad 3-z=0$

$x=2 \quad y=1 \quad z=3$

$D=(2,1,3)$



* Skalarni proizvod $\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1| \cdot |\vec{v}_2| \cos\varphi$ $\quad \varphi = \sphericalangle(\vec{v}_1, \vec{v}_2)$

- Za vektore u ravni: $\vec{v}_1 \cdot \vec{v}_2 = (x_1, y_1) \cdot (x_2, y_2) = x_1 x_2 + y_1 y_2$
- Za vektore u prostoru $\vec{v}_1 \cdot \vec{v}_2 = (x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = x_1 x_2 + y_1 y_2 + z_1 z_2$

* Vektorski proizvod

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

- Osobine vektorskog proizvoda:

a) Površina paralelograma nad $\vec{v}_1$ i $\vec{v}_2$ je:

$P = |\vec{v}_1 \times \vec{v}_2| = |\vec{v}_1||\vec{v}_2| \sin\varphi$

b) $\vec{v}_1 \times \vec{v}_2 \perp \vec{v}_1$ i $\vec{v}_1 \times \vec{v}_2 \perp \vec{v}_2$

* Mešoviti proizvod

$$(\vec{v}_1 \times \vec{v}_2) \cdot \vec{v}_3 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

- Osobine mešovitog proizvoda:

a) Zapremina paralelopipeda nad vektorima $\vec{v}_1, \vec{v}_2$ i $\vec{v}_3$ je

$V = |(\vec{v}_1 \times \vec{v}_2) \cdot \vec{v}_3|$

b) Vektori $\vec{v}_1, \vec{v}_2$ i $\vec{v}_3$ su komplanarni $\iff (\vec{v}_1 \times \vec{v}_2) \cdot \vec{v}_3 = 0$

14. a) $\vec{a} = (1, 2)$ $\quad \vec{a} \cdot \vec{b} = (1, 2) \cdot (-3, 6) = -3 + 12 = 9$

$\vec{b} = (-3, 6)$

b) $a = 2\vec{i} - 3\vec{j} = (2, -3)$ $\quad \vec{a} \cdot \vec{b} = (2, -3) \cdot (-4, 5) = -8 + (-15) = -23$

$\vec{b} = -4\vec{i} + 5\vec{j} = (-4, 5)$

15. a) $\vec{a}=(-1,1)$ $\quad \cos\varphi=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$\vec{b}=(2,0)$

$$\cos\varphi=\frac{(-1,1)(2,0)}{\sqrt{(-1)^2+1^2}\cdot\sqrt{2^2+0^2}}=\frac{-2+0}{2\sqrt{2}}=-\frac{\cancel{2}}{\cancel{2}\sqrt{2}}=-\frac{\sqrt{2}}{2}$$

$$\varphi=\frac{3\pi}{4}$$

b) $\vec{a}=2\vec{i}-5\vec{j}=(2,-5)$ $\quad \cos\varphi=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$\vec{b}=-\vec{i}+2\vec{j}=(-1,2)$

$$\cos\varphi=\frac{(2,-5)(-1,2)}{\sqrt{2^2+(-5)^2}\cdot\sqrt{(-1)^2+2^2}}=\frac{-2+(-10)}{\sqrt{29}\cdot\sqrt{5}}$$

$$\cos\varphi=\frac{-12}{\sqrt{145}}$$

16. $A(5,-1)$ $\quad \vec{OA}=(5,-1)$

$B(3,1)$ $\quad \vec{AB}=(3-5,1-(-1))=(-2,2)$

$$\vec{OA}\cdot\vec{AB}=(5,-1)\cdot(-2,2)=-10-2=-12$$

17. a) $\vec{a}=(2,1)$ $\quad \vec{a}\cdot\vec{b}=(2,1)(1,-1)=2-1=1$, vektori nisu ortogonalni

$\vec{b}=(1,-1)$ $\quad (2,1)=k(1,-1)$

$k=2$

$k=-1$ nisu ni kolinearni

b) $\vec{a}=(5,2)$ $\quad \vec{a}\cdot\vec{b}=(5,2)(4,-10)=20-20=0$ - vektori su ortogonalni

$\vec{b}=(4,-10)$

c) $\vec{a}=-\vec{i}+2\vec{j}=(-1,2)$ $\quad \vec{a}\cdot\vec{b}=(-1,2)(2,-4)=-2-8=-10$ - nisu ortogonalni

$\vec{b}=2\vec{i}-4\vec{j}=(2,-4)$ $\quad (2,-4)=k(-1,2)$

$k=-2$

$-4=2k$ $\quad$ vektori su kolinearni

$k=-2$

18. a) $\vec{a} = (2,-1)$
$\vec{b} = (\lambda, 4)$

i) $\vec{a}\cdot\vec{b} = 0$

$(2,-1)\cdot(\lambda,4) = 0$

$2\lambda - 4 = 0$

$2\lambda = 4$

$\lambda = 2$

ii) $\vec{b} = k\vec{a}$

$(\lambda,4) = k(2,-1)$

$\lambda = 2k$

$-k = 4$

$k = -4$

$\lambda = -8$

b) $\vec{a} = \lambda\vec{i} = (\lambda, 0)$
$\vec{b} = 2\vec{i} - \vec{j} = (2,-1)$

i) $\vec{a}\cdot\vec{b} = 0$

$(\lambda,0)(2,-1) = 0$

$2\lambda = 0$

$\lambda = 0$

ii) $\vec{b} = k\vec{a}$

$(2,-1) = k(\lambda,0)$

$2 = k\lambda$

$-1 \neq 0$

vektori nikad nisu kolinearni.

19. a) $\vec{a} = 3\vec{i} + 2\vec{j} - \vec{k} = (3,2,-1)$
$\vec{b} = -\vec{i} + 3\vec{j} - 2\vec{k} = (-1,3,-2)$

$\vec{a}\cdot\vec{b} = (3,2,-1)(-1,3,-2) = -3+6+2 = 5$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 2 & -1 \\ -1 & 3 & -2 \end{vmatrix} = \vec{i}\begin{vmatrix} 2 & -1 \\ 3 & -2 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & -1 \\ -1 & -2 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & 2 \\ -1 & 3 \end{vmatrix} =$$

$$= \vec{i}(-4+3) - \vec{j}(-6-1) + \vec{k}(9+2) = -\vec{i} + 7\vec{j} + 11\vec{k}$$

b) $\vec{a} = (3,0,-4)$
$\vec{b} = (1,2,0)$

$\vec{a}\cdot\vec{b} = (3,0,-4)\cdot(1,2,0) = 3+0-0 = 3$

$$\vec{a}\times\vec{b} = \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & -4 \\ 1 & 2 & 0 \end{matrix}\right|\begin{matrix} \vec{i} & \vec{j} \\ 3 & 0 \\ 1 & 2 \end{matrix} = \vec{i}\cdot 0\cdot 0 + \vec{j}\cdot(-4)\cdot 1 + \vec{k}\cdot 3\cdot 2 - \vec{k}\cdot 0\cdot 1 - \vec{i}(-4)\cdot 2 -$$

$$- \vec{j}\cdot 3\cdot 0 = -4\vec{j} + 6\vec{k} + 8\vec{i}$$

$$= 8\vec{i} - 4\vec{j} + 6\vec{k}$$

$$= 2(4,-2,3)$$

20. a) $\vec{a} = (-2, 1, 3)$

$\vec{b} = (p+1, -p, -3)$

$\vec{a} \cdot \vec{b} = 0$

$(-2, 1, 3) \cdot (p+1, -p, -3) = 0$

$-2p - 2 - p - 9 = 0$

$-3p = 11$

$p = -\frac{11}{3}$

b) $\vec{a} = 2\vec{i} + p\vec{j} + \vec{k} = (2, p, 1)$

$\vec{b} = 2p\vec{i} + 4\vec{j} - 2\vec{k} = (2p, 4, -2)$

$\vec{a} \cdot \vec{b} = 0$

$(2, p, 1) \cdot (2p, 4, -2) = 0$

$4p + 4p - 2 = 0$

$8p = 2$

$p = \frac{1}{4}$

21. $\vec{a} = (-1, 1, 0)$

$\vec{b} = (1, -2, -2)$

$\vec{d_1} = \vec{a} + \vec{b}$

$\vec{d_1} = (-1, 1, 0) + (1, -2, -2)$

$\vec{d_1} = (0, -1, -2)$

$|\vec{d_1}| = \sqrt{0 + (-1)^2 + (-2)^2} = \sqrt{1 + 4} = \sqrt{5}$

$\vec{b} + \vec{d_2} = \vec{a}$

$\vec{d_2} = \vec{a} - \vec{b}$

$\vec{d_2} = (-1, 1, 0) - (1, -2, -2)$

$\vec{d_2} = (-2, 3, 2)$

$|\vec{d_2}| = \sqrt{(-2)^2 + 3^2 + 2^2} = \sqrt{17}$

$$\cos\varphi = \frac{\vec{d_1} \cdot \vec{d_2}}{|\vec{d_1}||\vec{d_2}|} = \frac{(0, -1, -2) \cdot (-2, 3, 2)}{\sqrt{5} \cdot \sqrt{17}} = \frac{-3 - 4}{\sqrt{85}} = \frac{-7}{\sqrt{85}}$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ 1 & -2 & -2 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & 0 \\ -2 & -2 \end{vmatrix} - \vec{j} \begin{vmatrix} -1 & 0 \\ 1 & -2 \end{vmatrix} + \vec{k} \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} =$$

$$= \vec{i}(-2 + 0) - \vec{j}(2 - 0) + \vec{k}(2 - 1) = -2\vec{i} - 2\vec{j} + \vec{k}$$

$$P = |\vec{a} \times \vec{b}| = \sqrt{4+4+1} = \sqrt{9} = 3$$

22. a) $\vec{a} = 6\vec{i} + \vec{j} + \vec{k} = (6, 1, 1)$

$\vec{b} = 3\vec{j} - \vec{k} = (0, 3, -1)$

$\vec{c} = -2\vec{i} + 3\vec{j} + 5\vec{k} = (-2, 3, 5)$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} 6 & 1 & 1 \\ 0 & 3 & -1 \\ -2 & 3 & 5 \end{vmatrix} \begin{matrix} 6 & 1 \\ 0 & 3 \\ -2 & 3 \end{matrix} = 90 + 2 + 6 + 18 = 116 \neq 0$$

Vektori nisu komplanarni.

$\vec{d} = \vec{a} + \lambda \vec{b}$

$\vec{d} = (6, 1, 1) + \lambda(0, 3, -1) = (6, 1 + 3\lambda, 1 - \lambda)$

$\vec{d} \cdot \vec{c} = 0$

$(6, 1 + 3\lambda, 1 - \lambda) \cdot (-2, 3, 5) = 0$

$-12 + 3 + 9\lambda + 5 - 5\lambda = 0$

$4\lambda - 4 = 0$

$4\lambda = 4$

$\lambda = 1$

b) $\vec{a} = (2, 0, -1)$

$\vec{b} = (-1, 3, 0)$

$\vec{c} = (5, -3, -2)$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} 2 & 0 & -1 \\ -1 & 3 & 0 \\ 5 & -3 & -2 \end{vmatrix} \begin{matrix} 2 & 0 \\ -1 & 3 \\ 5 & -3 \end{matrix} = -12 - 3 + 15 = 0$$

Vektori su ortogonalni.

$\vec{d} = \vec{a} + \lambda \vec{b} = (2, 0, -1) + \lambda(-1, 3, 0) = (2 - \lambda, 3\lambda, -1)$

$\vec{d} \cdot \vec{c} = 0$

$(2 - \lambda, 3\lambda, -1) \cdot (5, -3, -2) = 0$

$10 - 5\lambda - 9\lambda + 2 = 0$

$-14\lambda = -12$

$\lambda = \frac{12}{14} = \frac{6}{7}$

23. a) $A(-3,-2,0)$ $\quad \vec{AB}=(6,-1,1)$

$B(3,-3,1)$ $\quad \vec{AC}=(8,2,2)$

$C(5,0,2)$

$$\vec{AB}\times\vec{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 6 & -1 & 1\\ 8 & 2 & 2\end{vmatrix}\begin{matrix}\vec{i} & \vec{j}\\ 6 & -1\\ 8 & 2\end{matrix} = -2\vec{i}+8\vec{j}+12\vec{k}+$$

$$+8\vec{k}-2\vec{i}-12\vec{j}=$$

$$=-4\vec{i}-4\vec{j}+20\vec{k}=$$

$$=4(-1,-1,5)$$

$$P=\frac{1}{2}\cdot 4\sqrt{1+1+25}=2\cdot\sqrt{27}=6\sqrt{3}$$

b) $A(0,0,1)$ $\quad \vec{AB}=(4,0,0)$ $\quad c=|\vec{AB}|=\sqrt{16}=4 \Rightarrow D=(2,0,1)$

$B(4,0,1)$ $\quad \vec{AC}=(2,1,0)$ $\quad |\vec{AC}|=\sqrt{5} \Rightarrow$ $\quad \vec{CD}=(0,-1,0)$

$C(2,1,1)$ $\quad \vec{BC}=(-2,1,0)$ $\quad |\vec{BC}|=\sqrt{5} \Rightarrow$ $\quad t_c=h_c=\sqrt{1}=1$

$$P=\frac{c\cdot h_c}{2}=\frac{4\cdot 1}{2}=2$$

24. $A(1,0,1)$ $\quad \vec{AB}=(1,-1,-1)$

$B(2,-1,0)$ $\quad \vec{AC}=(-2,1,-1)$

$C(-1,1,0)$

$$\vec{a}=\vec{AB}\times\vec{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -1 & -1\\ -2 & 1 & -1\end{vmatrix}\begin{matrix}\vec{i} & \vec{j}\\ 1 & -1\\ -2 & 1\end{matrix} = \vec{i}+2\vec{j}+\vec{k}-2\vec{k}+\vec{i}+\vec{j}=$$

$$=2\vec{i}+3\vec{j}-\vec{k}$$

$$\vec{a}_0=\pm\frac{\vec{a}}{|\vec{a}|}=\pm\frac{(2,3,-1)}{\sqrt{14}}=\pm\left(\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}},\frac{-1}{\sqrt{14}}\right)$$

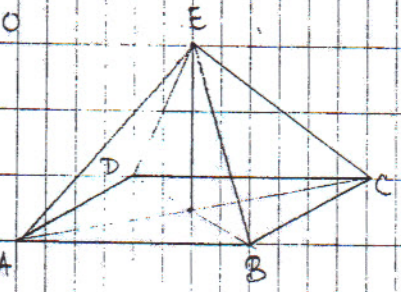
25. $A(1,0,1)$ $\quad \vec{AB}=\vec{DC}$

$B(3,1,1)$ $\quad (2,1,0)=(4-x,2-y,3-z)$

$C(4,2,3)$ $\quad 4-x=2 \quad 2-y=1 \quad 3-z=0$

$E(2,2,6)$ $\quad x=2 \quad y=1 \quad z=3$

$D(2,1,3)$



$$\vec{AB}\times\vec{AD} = \left|\begin{array}{ccc|cc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 2 & 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 \end{array}\right| = 2\vec{i} + 2\vec{k} - \vec{k} - 4\vec{j} = 2\vec{i} - 4\vec{j} + \vec{k}$$

$$|\vec{AB}\times\vec{AD}| = \sqrt{4+16+1} = \sqrt{21} = B$$

$$V_1 = (\vec{AB}\times\vec{AD})\cdot AE = \left|\begin{array}{ccc|cc} 2 & 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 5 & 1 & 2 \end{array}\right| = 10 + 2 - 8 - 5 = -1$$

$V_1 = 1$

$V = \frac{1}{3}V_1$

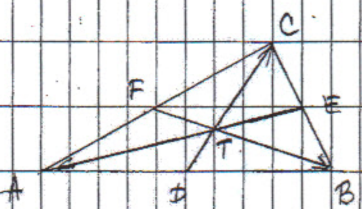
$V = \frac{1}{3}$

$V = \frac{1}{3}BH$

$H = \frac{3V}{B} = \frac{1}{\sqrt{21}}$

5.3. Zadaci za vežbu

1. $\vec{TA} + \vec{TB} + \vec{TC} = 0$



$\vec{TA} = \vec{TD} + \vec{DA}$

$\vec{TB} = \vec{TD} + \vec{DB}$

$\vec{TC} : \vec{DT} = 2 : 1$

$\vec{TC} = 2\vec{DT}$

$\vec{TA} + \vec{TB} + \vec{TC} =$

$= \vec{TD} + \vec{DA} + \vec{TD} + DB + 2\vec{DT}$

$\vec{DA} = -\vec{DB}$

$\vec{TA} + \vec{TB} + \vec{TC} = 2\vec{TD} - \cancel{\vec{DB}} + \cancel{\vec{DB}} + 2\vec{DT}$

$\vec{TA} + \vec{TB} + \vec{TC} = 2\vec{TD} + 2\vec{DT}$ $\qquad \vec{TD} = -\vec{DT}$

$\vec{TA} + \vec{TB} + \vec{TC} = -2\vec{DT} + 2\vec{DT}$

$\vec{TA} + \vec{TB} + \vec{TC} = 0$

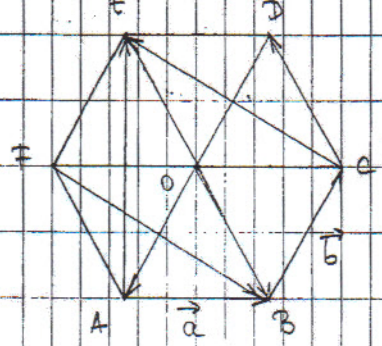
2. $\vec{AB} = \vec{a}$

$\vec{BC} = \vec{b}$

$\vec{OD} = \vec{BC} = \vec{b}$

$\vec{CO} = -\vec{AB} = -\vec{a}$

$\vec{CD} = \vec{CO} + \vec{OD} = -\vec{a} + \vec{b} = \vec{b} - \vec{a}$



$\vec{BO} = \vec{OE} = \vec{CD} = \vec{b} - \vec{a}$

$\vec{AE} = \vec{AB} + \vec{BE}$

$\vec{AE} = \vec{a} + 2(\vec{b} - \vec{a})$

$\vec{AE} = \vec{a} + 2\vec{b} - 2\vec{a}$

$\vec{AE} = 2\vec{b} - \vec{a}$

$\vec{FE} = \vec{BC} = \vec{b}$

$\vec{FB} = \vec{FE} + \vec{EB}$

$\vec{FB} = \vec{FE} - \vec{BE}$

$\vec{FB} = \vec{b} - 2(\vec{b} - \vec{a})$

$\vec{FB} = \vec{b} - 2\vec{b} + 2\vec{a}$

$\vec{FB} = 2\vec{a} - \vec{b}$

$\vec{OA} = -\vec{BC}$

$\vec{OA} = -\vec{b}$

$\vec{OB} = -\vec{CD}$

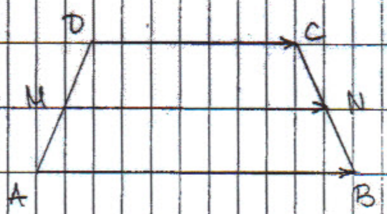
$\vec{OB} = \vec{a} - \vec{b}$

$\vec{CE} = -\vec{FB}$

$\vec{CE} = \vec{b} - 2\vec{a}$

$$\begin{aligned} 3\vec{OB} - \frac{1}{2}\vec{CE} &= 3(\vec{a} - \vec{b}) - \frac{1}{2}(\vec{b} - 2\vec{a}) \\ &= 3\vec{a} - 3\vec{b} - \frac{1}{2}\vec{b} + \vec{a} \\ &= 4\vec{a} - \frac{7}{2}\vec{b} \end{aligned}$$

3. $2\vec{MN} = \vec{AB} + \vec{DC}$



$\vec{MN} = \vec{MD} + \vec{DC} + \vec{CN}$

$\vec{MN} = \vec{MA} + \vec{AB} + \vec{BN}$

$2\vec{MN} = \vec{MD} + \vec{DC} + \vec{CN} + \vec{MA} + \vec{AB} + \vec{BN}$

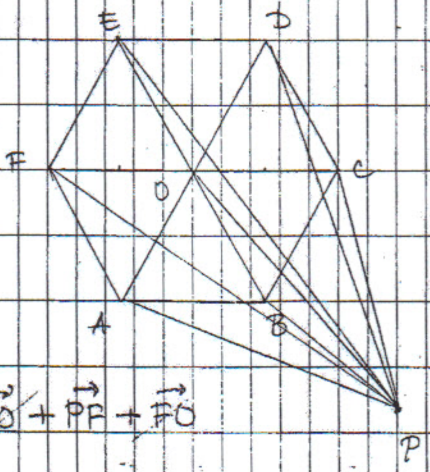
$2\vec{MN} = \cancel{-\vec{MA}} + \vec{DC} \cancel{- \vec{BN}} + \cancel{\vec{MA}} + \vec{AB} + \cancel{\vec{BN}}$ $\qquad \vec{MD} = -\vec{MA}$

$2\vec{MN} = \vec{AB} + \vec{DC}$ $\qquad \vec{CN} = -\vec{BN}$

4. $\vec{PO} = \vec{PA} + \vec{AO}$ $\qquad \vec{AO} = -\vec{DO}$

$\vec{PO} = \vec{PB} + \vec{BO}$ $\qquad \vec{BO} = -\vec{EO}$

$\vec{PO} = \vec{PC} + \vec{CO}$ $\qquad \vec{CO} = -\vec{FO}$



$\vec{PO} = \vec{PD} + \vec{DO}$

$\vec{PO} = \vec{PE} + \vec{EO}$

$\vec{PO} = \vec{PF} + \vec{FO}$

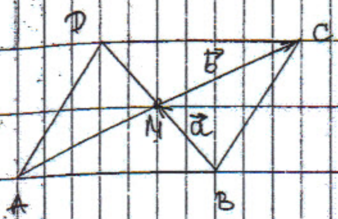
$6\vec{PO} = \vec{PA} \cancel{- \vec{DO}} + \vec{PB} \cancel{- \vec{EO}} + \vec{PC} - \vec{FO} + \vec{PD} + \cancel{\vec{DO}} + \vec{PE} + \cancel{\vec{EO}} + \vec{PF} + \vec{FO}$

$6\vec{PO} = \vec{PA} + \vec{PB} + \vec{PC} + \vec{PD} + \vec{PE} - \vec{PF}$

$\vec{PO} = \frac{1}{6}(\vec{PA} + \vec{PB} + \vec{PC} + \vec{PD} + \vec{PE} + \vec{PF})$

5. $\vec{a}=\vec{BM}$ $\quad \vec{BC}=\vec{BM}+\vec{MC}$ $\quad \vec{BM}=\vec{MD}$

$\vec{b}=\vec{MC}$ $\quad \vec{BC}=\vec{AD}=\vec{a}+\vec{b}$ $\quad \vec{DM}=-\vec{BM}=-\vec{a}$



$\vec{DA}=-\vec{AD}$ $\quad \vec{DC}=\vec{DM}+\vec{MC}$

$\vec{DA}=-\vec{a}-\vec{b}$ $\quad \vec{DC}=-\vec{a}+\vec{b}=\vec{b}-\vec{a}$ $\quad \vec{CD}=\vec{a}-\vec{b}$

$$\frac{1}{4}\vec{DA}-2\vec{CD}=\frac{1}{4}(-\vec{a}-\vec{b})-2(\vec{a}-\vec{b})$$
$$=-\frac{1}{4}\vec{a}-\frac{1}{4}\vec{b}-2\vec{a}+2\vec{b}$$
$$=-\frac{1}{4}\vec{a}-\frac{1}{4}\vec{b}-\frac{8}{4}\vec{a}+\frac{8}{4}\vec{b}$$
$$=\frac{7\vec{b}-9\vec{a}}{4}$$

6. a) $\vec{a}=(2,-1,2)$ $\quad |\vec{a}|=\sqrt{4+1+4}=\sqrt{9}=3$

$\vec{b}=(0,0,3)$ $\quad 2\vec{b}+3\vec{a}=2(0,0,3)+3(2,-1,2)=$
$$=(0,0,6)+(6,-3,6)=$$
$$=(6,-3,12)$$

b) $\vec{a}=\vec{i}+2\vec{j}=(1,2,0)$ $\quad |\vec{a}|=\sqrt{1+4}=\sqrt{5}$

$\vec{b}=-\vec{j}+3\vec{k}=(0,-1,3)$ $\quad 2\vec{b}+3\vec{a}=2(0,-1,3)+3(1,2,0)=$
$$=(0,-2,6)+(3,6,0)=$$
$$=(3,4,6)$$

7. a) $\vec{a}=(p-2)\vec{i}+3\vec{j}+(p-1)\vec{k}$

$\vec{b}=\vec{i}+3\vec{j}+2\vec{k}$

$|\vec{a}|=|\vec{b}|$

$\sqrt{(p-2)^2+9+(p-1)^2}=\sqrt{1+9+4}$

$p^2-4p+4+9+p^2-2p+1=14$

$2p^2-6p=0$

$2p(p-3)=0$

$p=0 \quad \vee \quad p-3=0 \quad p=3 \rightarrow$ isti intenzitet vektora

$\vec{b}=k\vec{a}$

$(1,3,2)=k(p-2,3,p-1)$

$kp-2k=1$

$3=3k,\ k=1$

$kp-k=2$

$p=3$

$\bar{E}$ b) $\vec{a} = (1, 0, p+1)$ $\quad |\vec{a}| = |\vec{b}|$ $\quad \vec{b} = k\vec{a}$

$\bar{A}$ $\vec{b} = (p, 0, 6)$ $\quad \sqrt{1+(p+1)^2} = \sqrt{p^2+36}$ $\quad (p, 0, 6) = k(1, 0, p+1)$

$\bar{A}$ $1 + p^2 + 2p + 1 = p^2 + 36$ $\quad k = p$

$\bar{A}$ $2p = 34$ $\quad kp + k = 6$

$\bar{A}$ $p = 17$ $\quad p^2 + p - 6 = 0$

$$p_{1,2} = \frac{-1 \pm \sqrt{1+24}}{2}$$

$$p_1 = 2, \quad p_2 = -3$$

8\. a) $\vec{a} = (-1, 2)$ $\quad \vec{a} \cdot \vec{b} = (-1, 2) \cdot (5, -4)$

$\vec{b} = (5, -4)$ $\quad = -5 + (-8) = -13$

3 b) $\vec{a} = 5\vec{i} = (5, 0)$ $\quad \vec{a} \cdot \vec{b} = (5, 0) \cdot (-3, 2)$

$\vec{b} = -3\vec{i} + 2\vec{j} = (-3, 2)$ $\quad = -15 + 0 = -15$

9 a) $\vec{a} = (5, -1)$ $\quad \cos\varphi = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$\vec{b} = (0, 3)$

$$\cos\varphi = \frac{(5,-1)(0,3)}{\sqrt{25+1}\,\sqrt{9}} = \frac{-\cancel{3}\,1}{\sqrt{26} \cdot \cancel{3}}$$

4 $$\cos\varphi = -\frac{1}{\sqrt{26}}$$

b) $\vec{a} = -2\vec{i} + \vec{j} = (-2, 1)$ $\quad \cos\varphi = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$

$\vec{b} = 4\vec{i} - 2\vec{j} = (4, -2)$

$$\cos\varphi = \frac{(-2,1)(4,-2)}{\sqrt{4+1}\,\sqrt{16+4}} = \frac{-8-2}{\sqrt{5}\,\sqrt{20}}$$

$$= \frac{-10}{\sqrt{100}} = -\frac{10}{10} = -1$$

10\. a) $A(-1, -2)$ $\quad \vec{OA} = (-1, -2)$ $\quad \vec{OA} \cdot \vec{AB} = (-1, -2) \cdot (4, 8)$

$B(3, 6)$ $\quad \vec{AB} = (4, 8)$ $\quad = -4 - 16$

$= -20$

b) $A(-1,1)$ $\quad \vec{OA}=(-1,1)$ $\quad \vec{OA}\cdot\vec{AB}=(-1,1)\cdot(3,1)=-3+1=-2$

$B(2,2)$ $\quad \vec{AB}=(3,1)$

11. a) $\vec{a}=3\vec{i}-\vec{j}=(3,-1)$ $\quad \vec{a}\cdot\vec{b}=(3,-1)\cdot(1,3)$

$\vec{b}=\vec{i}+3\vec{j}=(1,3)$ $\quad =3-3=0$ - vektori su ortogonalni

b) $\vec{a}=(-1,-1)$ $\quad \vec{a}\cdot\vec{b}=(-1,-1)\cdot(0,2)=0-2=-2$ - nisu ortogonalni

$\vec{b}=(0,2)$

$\vec{b}=k\vec{a}$

$(0,2)=k(-1,-1)$

$k=0$

$k=-2$ nisu kolinearni

c) $\vec{a}=(6,-3)$ $\quad \vec{a}\cdot\vec{b}=(6,-3)(-2,1)=-12-3=-15$ - nisu ortogonalni

$\vec{b}=(-2,1)$

$\vec{b}=k\vec{a}$

$(-2,1)=k(6,-3)$

$6k=-2$ $\quad -3k=1$

$k=-\frac{1}{3}$ $\quad k=-\frac{1}{3}$ - kolinearni su

12. a) $\vec{a}=3\vec{i}-2\vec{j}=(3,-2)$ $\quad$ i) $\vec{a}\cdot\vec{b}=0$ $\quad$ ii) $\vec{b}=k\vec{a}$

$\vec{b}=6\vec{i}+\lambda\vec{j}=(6,\lambda)$

$(3,-2)(6,\lambda)=0$ $\quad (6,\lambda)=k(3,-2)$

$18-2\lambda=0$ $\quad 3k=6$ $\quad \lambda=-2k$

$2\lambda=18$ $\quad k=2$ $\quad \lambda=-4$

$\lambda=9$

b) $\vec{a}=(5,4)$ $\quad$ i) $\vec{a}\cdot\vec{b}=0$ $\quad$ ii) $\vec{b}=k\vec{a}$

$\vec{b}=(\lambda,-2)$

$(5,4)(\lambda,-2)=0$ $\quad (\lambda,-2)=k(5,4)$

$5\lambda-8=0$ $\quad 5k=\lambda$ $\quad 4k=-2$

$5\lambda=8$ $\quad \lambda=-\frac{5}{2}$ $\quad k=-\frac{1}{2}$

$\lambda=\frac{8}{5}$

13. a) $\vec{a} = (2,-1,2)$

$\vec{b} = (0,0,3)$

i) $\vec{a} \cdot \vec{b} = (2,-1,2)(0,0,3) =$

$= 6$

ii) $\vec{a} \times \vec{b} = \left| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 2 \\ 0 & 0 & 3 \end{matrix} \right| \begin{matrix} \vec{i} & \vec{j} \\ 2 & -1 \\ 0 & 0 \end{matrix} =$

$= -3\vec{i} - 6\vec{j}$

$= 3(-1,-2,0)$

b) $\vec{a} = \vec{i} + 2\vec{j} = (1,2,0)$

$\vec{b} = -\vec{j} + 3\vec{k} = (0,-1,3)$

$\vec{a} \cdot \vec{b} = (1,2,0) \cdot (0,-1,3) =$

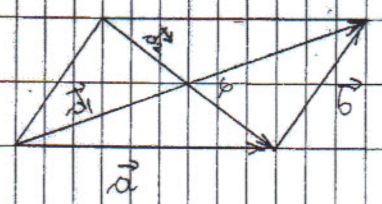
$= -2$

$\vec{a} \times \vec{b} = \left| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 0 & -1 & 3 \end{matrix} \right| \begin{matrix} \vec{i} & \vec{j} \\ 1 & 2 \\ 0 & -1 \end{matrix} = 6\vec{i} - 1\vec{k} - 3\vec{j} = 6\vec{i} - 3\vec{j} - k = (6,-3,-1)$

3

14. a) $\vec{a} = \vec{i} + 2\vec{j} = (1,2,0)$

$\vec{b} = -2\vec{j} + \vec{k} = (0,-2,1)$



$\vec{d_1} = \vec{a} + \vec{b} = (1,0,1)$

$|\vec{d_1}| = \sqrt{1+1} = \sqrt{2}$

$\vec{d_2} = -\vec{b} + \vec{a} = \vec{a} - \vec{b} = (1,4,-1)$

4 $|\vec{d_2}| = \sqrt{1+16+1} = \sqrt{18} = 3\sqrt{2}$

$\cos\varphi = \dfrac{\vec{d_1} \cdot \vec{d_2}}{|\vec{d_1}||\vec{d_2}|} = \dfrac{(1,0,1)(1,4,-1)}{\sqrt{2} \cdot 3\sqrt{2}}$

$\cos\varphi = \dfrac{0}{6} = 0$

$\varphi = 90^\circ = \dfrac{\pi}{2}$

$\vec{a} \times \vec{b} = \left| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 0 & -2 & 1 \end{matrix} \right| \begin{matrix} \vec{i} & \vec{j} \\ 1 & 2 \\ 0 & -2 \end{matrix} = 2\vec{i} - 2\vec{k} - \vec{j} = 2\vec{i} - \vec{j} - 2\vec{k} = (2,-1,-2)$

$P = |\vec{a} \times \vec{b}| = \sqrt{4+1+4} = \sqrt{9} = 3$

b) $\vec{a} = (1,0,0)$

$\vec{b} = (3,0,-2)$

$\vec{d_1} = \vec{a} + \vec{b} = (4,0,-2)$

$|\vec{d_1}| = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$

$\vec{d_2} = -\vec{b} + \vec{a} = \vec{a} - \vec{b} = (-2,0,2)$

$|\vec{d_2}| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$\cos\varphi = \dfrac{\vec{d_1} \cdot \vec{d_2}}{|\vec{d_1}||\vec{d_2}|} = \dfrac{(4,0,-2)\,(-2,0,2)}{2\sqrt{5} \cdot 2\sqrt{2}} =$

$= \dfrac{-8-4}{4\sqrt{10}} = \dfrac{-12}{4\sqrt{10}} = -\dfrac{3}{\sqrt{10}}$

$$\vec{a}\times\vec{b}=\left|\begin{array}{ccc|cc}\vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j}\\ 1 & 0 & 0 & 1 & 0\\ 3 & 0 & -2 & 3 & 0\end{array}\right| = 2\vec{j}=(0,2,0)$$

$$P=|\vec{a}\times\vec{b}|=\sqrt{4}=2$$

15. a) $\vec{a}=-\vec{i}+2\vec{j}=(-1,2,0)$

$\vec{b}=3\vec{i}+2\vec{j}-\vec{k}=(3,2,-1)$

$\vec{c}=4\vec{i}-\vec{k}=(4,0,-1)$

$$(\vec{a}\times\vec{b})\cdot\vec{c}=\left|\begin{array}{ccc|cc}-1 & 2 & 0 & -1 & 2\\ 3 & 2 & -1 & 3 & 2\\ 4 & 0 & -1 & 4 & 0\end{array}\right| = 2-8+6=0$$ - vektori su komplanarni

$\vec{a}+\lambda\vec{b}=(-1,2,0)+\lambda(3,2,-1)=(3\lambda-1,\ 2\lambda+2,\ -\lambda)$

$(\vec{a}+\lambda\vec{b})\cdot\vec{c}=0$

$(3\lambda-1,\ 2\lambda+2,\ -\lambda)\cdot(4,0,-1)=0$

$12\lambda-4+\lambda=0$

$13\lambda=4$

$\lambda=\frac{4}{13}$

b) $\vec{a}=(0,0,2)$

$\vec{b}=(-1,3,2)$

$\vec{c}=(2,0,-1)$

$$(\vec{a}\times\vec{b})\cdot\vec{c}=\left|\begin{array}{ccc|cc}0 & 0 & 2 & 0 & 0\\ -1 & 3 & 2 & -1 & 3\\ 2 & 0 & -1 & 2 & 0\end{array}\right| = -12\neq 0$$

Vektori nisu komplanarni

$\vec{a}+\lambda\vec{b}=(0,0,2)+\lambda(-1,3,2)=(-\lambda,\ 3\lambda,\ 2\lambda+2)$

$(\vec{a}+\lambda\vec{b})\cdot\vec{c}=0$

$(-\lambda,\ 3\lambda,\ 2\lambda+2)\cdot(2,0,-1)=0$

$-2\lambda-2\lambda-2=0$

$-4\lambda=2$

$\lambda=-\frac{1}{2}$

16. a) $A(-2,1,0)$ $\quad \vec{AB}=(3,0,0)$

$B(1,1,0)$ $\quad \vec{AC}=(5,1,0)$

$C(3,2,0)$

$$\vec{a}=\vec{AB}\times\vec{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & 0 & 0\\ 5 & 1 & 0\end{vmatrix}\begin{matrix}\vec{i} & \vec{j}\\ 3 & 0\\ 5 & 1\end{matrix}=3\vec{k}=(0,0,3)$$

$$\vec{a}_0=\pm\frac{\vec{a}}{|\vec{a}|}=\pm\frac{(0,0,3)}{\sqrt{9}}=\pm(0,0,1)$$

b) $A(0,1,-1)$ $\quad \vec{AB}=(2,-1,-2)$

$B(2,0,-3)$ $\quad \vec{AC}=(1,-1,0)$

$C(1,0,-1)$

$$\vec{a}=\vec{AB}\times\vec{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -1 & -2\\ 1 & -1 & 0\end{vmatrix}\begin{matrix}\vec{i} & \vec{j}\\ 2 & -1\\ 1 & -1\end{matrix}=-2\vec{j}-2\vec{k}+\vec{k}-2\vec{i}=$$

$$=-2\vec{i}-2\vec{j}-\vec{k}=$$

$$=(-2,-2,-1)$$

$$\vec{a}_0=\pm\frac{\vec{a}}{|\vec{a}|}=\pm\frac{(-2,-2,-1)}{\sqrt{9}}=\pm\left(-\frac{2}{3},-\frac{2}{3},-\frac{1}{3}\right)$$

17. a) $A(-5,0,3)$ $\quad \vec{AB}=(6,1,-2)$ $\quad D=\left(\frac{-5+2}{2},\frac{0+0}{2},\frac{3-1}{2}\right)$

$B(1,1,1)$ $\quad |\vec{AB}|=\sqrt{36+1+4}=\sqrt{41}$ $\quad D=\left(-\frac{3}{2},0,1\right)$

$C(2,0,-1)$ $\quad \vec{BC}=(1,-1,-2)$

$|\vec{BC}|=\sqrt{1+1+4}=\sqrt{6}$ $\quad \vec{t}_b=\vec{BD}=\left(-\frac{5}{2},-1,0\right)$

$\vec{AC}=(7,0,-4)$

$b=|\vec{AC}|=\sqrt{49+16}=\sqrt{65}$ $\quad |\vec{t}_b|=\sqrt{\frac{25}{4}+\frac{4}{4}}=\frac{\sqrt{29}}{2}$

b) $A(1,0,2)$ $\quad \vec{AB}=(0,6,0)$ $\quad D=\left(\frac{1+1}{2},\frac{6+0}{2},\frac{2+2}{2}\right)=(1,3,2)$

$B(1,6,2)$ $\quad c=|\vec{AB}|=\sqrt{36}=6$ $\quad \vec{t}_c=|\vec{BD}|=(0,-3,0)$

$C(1,3,-1)$ $\quad \vec{AC}=(0,3,-3)$ $\quad |\vec{t}_c|=\sqrt{9}=3$

$|\vec{AC}|=\sqrt{9+9}=\sqrt{18}=3\sqrt{2}$

$\vec{BC}=(0,-3,-3)$

$|\vec{BC}|=\sqrt{18}=3\sqrt{2}$

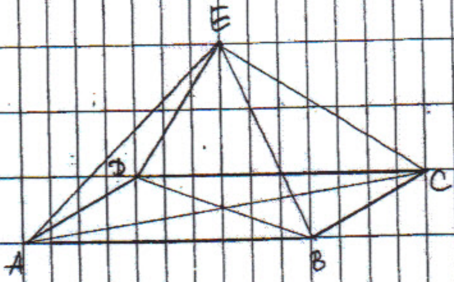
18 a) $A(4,0,1)$ $\quad \vec{AD} = \vec{BC}$

$B(2,1,1)$ $\quad (x-4, y, z-1) = (-2,-1,0)$

$C(0,0,1)$ $\quad x-4=-2 \quad y=-1 \quad z-1=0$

$E(2,3,7)$ $\quad x=2 \qquad z=1$

$D(2,-1,1)$



$$\vec{AB}\times\vec{AD} = \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & 0 \\ -2 & -1 & 0 \end{matrix}\right| \begin{matrix} \vec{i} & \vec{j} \\ -2 & 1 \\ -2 & -1 \end{matrix} = 2\vec{k} + 2\vec{k} = 4k = (0,0,4)$$

$$B = |\vec{AB}\times\vec{AD}| = \sqrt{16} = 4$$

$$(\vec{AB}\times\vec{AD})\times\vec{AE} = (0,0,4)\cdot(-2,3,6) = 24$$

$$V_1 = 24$$

$$V = \frac{1}{3}V_1 = 8$$

$$V = \frac{1}{3}BH$$

$$H = \frac{3V}{B} = \frac{3\cdot\cancel{8}^{2}}{\cancel{4}} = 6$$

b) $A(-1,-3,1)$ $\quad \vec{AD} = \vec{BC}$

$B(-1,-6,-2)$ $\quad (x+1, y+3, z-1) = (0,6,0)$

$C(-1,0,-2)$ $\quad x+1=0 \quad y+3=6 \quad z-1=0$

$E(0,4,1)$ $\quad x=-1 \quad y=3 \quad z=1$

2) $\quad D(-1,3,1)$

$$\vec{AB}\times\vec{AD} = \left|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -3 & -3 \\ 0 & 6 & 0 \end{matrix}\right| \begin{matrix} \vec{i} & \vec{j} \\ 0 & -3 \\ 0 & 6 \end{matrix} = 18\vec{i} = (18,0,0)$$

$$B = |\vec{AB}\times\vec{AD}| = 18$$

$$(\vec{AB}\times\vec{AD})\cdot\vec{AE} = (18,0,0)\cdot(1,7,0) = 18$$

$$V_1 = 18$$

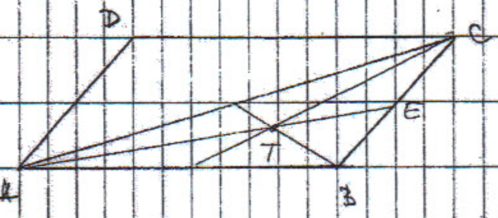
$V=\frac{1}{3}V_1$ $\qquad$ $V=\frac{1}{3}BH$

$V=6$ $\qquad$ $H=\frac{3V}{B}=\frac{3\cdot 6}{18}=1$

19. $A(1,1,2)$

$B(1,2,3)$

$C(-1,1,2)$



$\vec{AD}=\vec{BC}$

$(x-1,\ y-1,\ z-2)=(-2,-1,-1)$

$x-1=-2 \qquad y-1=-1 \qquad z-2=-1$

$x=-1 \qquad y=0 \qquad z=1$

3 $D(-1,0,1)$

$E=\left(\frac{-1+1}{2},\ \frac{2+1}{2},\ \frac{3+2}{2}\right)$

$E=\left(0,\ \frac{3}{2},\ \frac{5}{2}\right)$

$\vec{AT}:\vec{TE}=2:1$

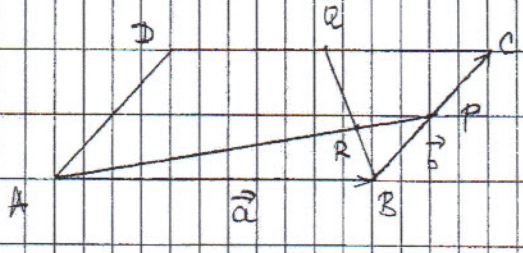
$\vec{r}_T=\frac{\vec{r}_A+2\vec{r}_E}{1+2}$

$\vec{r}_T=\frac{(1,1,2)+(0,3,5)}{3}$

$\vec{r}_T=\frac{(1,4,7)}{3}$

$T=\left(\frac{1}{3},\ \frac{4}{3},\ \frac{7}{3}\right)$

4 21.



a) $\varphi\vec{a}+\psi\vec{b}=0$

$\vec{a}$ i $\vec{b}$ su nekolinearni $\Rightarrow$

$\Rightarrow \varphi=0$ i $\psi=0$.

b) $\vec{PR}=\alpha\vec{PA}$

$\vec{RB}=\beta\vec{QB}$

$\vec{BP}+\vec{PC}=\vec{BC}$

$\vec{BP}=\vec{PC}$

$2\vec{BP}=\vec{BC}$

$\vec{BP}=\frac{1}{2}\vec{BC}$

$\vec{BP}=\frac{1}{2}\vec{b}$

$\vec{PA} = \vec{PB} + \vec{BA}$ $\qquad \vec{PB} = -\vec{BP}$ $\qquad \vec{QB} = \vec{QC} + \vec{CB}$ $\qquad \vec{QC} = \frac{1}{2}\vec{DC} = \frac{1}{2}\vec{AB} = \frac{1}{2}\vec{a}$

$\vec{PA} = -\frac{1}{2}\vec{b} - \vec{a}$ $\qquad \vec{PB} = -\frac{1}{2}\vec{b}$ $\qquad \vec{QB} = \frac{1}{2}\vec{a} - \vec{b}$ $\qquad \vec{CB} = -\vec{BC} = -\vec{b}$

$\alpha\vec{PA} = \alpha\left(-\frac{1}{2}\vec{b} - \vec{a}\right)$ $\qquad \vec{BA} = -\vec{AB}$ $\qquad \beta\vec{QB} = \beta\left(\frac{1}{2}\vec{a} - \vec{b}\right)$

$\vec{PR} = \alpha\left(-\frac{1}{2}\vec{b} - \vec{a}\right)$ $\qquad \vec{BA} = -\vec{a}$ $\qquad \vec{RB} = \beta\left(\frac{1}{2}\vec{a} - \vec{b}\right)$

c) $\vec{BP} + \vec{PR} + \vec{RB} = 0$

$\frac{1}{2}\vec{b} + \alpha\left(-\frac{1}{2}\vec{b} - \vec{a}\right) + \beta\left(\frac{1}{2}\vec{a} - \vec{b}\right) = 0$

$\frac{1}{2}\vec{b} - \frac{1}{2}\vec{b}\alpha - \vec{a}\alpha + \frac{1}{2}\vec{a}\beta - \vec{b}\beta = 0$

$\vec{b}\left(\frac{1}{2} - \frac{1}{2}\alpha - \beta\right) + \vec{a}\left(\frac{1}{2}\beta - \alpha\right) = 0$

$\frac{1}{2} - \frac{1}{2}\alpha - \beta = 0 \quad | \cdot 2$

$\frac{1}{2}\beta - \alpha = 0 \quad | \cdot 4$

$1 - \alpha - 2\beta = 0$

$2\beta - 4\alpha = 0$

$1 - 5\alpha = 0$

$5\alpha = 1$

$\alpha = \frac{1}{5}$

$\beta = 2\alpha = \frac{2}{5}$

$\vec{AR} + \vec{RP} = \vec{AP}$ $\qquad \vec{RP} = -\vec{PR}$

$\vec{AR} - \vec{PR} = -\vec{PA}$ $\qquad \vec{AP} = -\vec{PA}$

$\vec{AR} - \frac{1}{5}\left(-\frac{1}{2}\vec{b} - \vec{a}\right) = -\left(-\frac{1}{2}\vec{b} - \vec{a}\right)$

$\vec{AR} = \frac{1}{5}\left(-\frac{1}{2}\vec{b} - \vec{a}\right) - \left(-\frac{1}{2}\vec{b} - \vec{a}\right)$

$\vec{AR} = -\frac{4}{5}\left(-\frac{1}{2}\vec{b} - \vec{a}\right)$

$\vec{AR} = -\frac{4}{5}\vec{PA}$

$\vec{AR} = \frac{4}{5}\vec{AP}$

$\vec{PR} = \frac{1}{5}\vec{PA}$

$\vec{RP} = \frac{1}{5}\vec{AP}$

$\vec{AR} : \vec{RP} = \frac{4}{5} : \frac{1}{5}$

$\vec{AR} : \vec{RP} = 4 : 1$

24 $\quad A(0,0,0)$

$B(1,0,0)$

$C(1,1,0)$

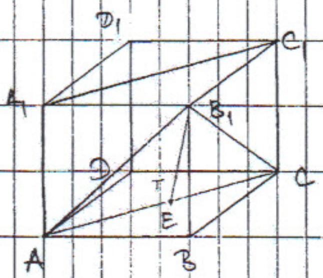
$D(0,1,0)$

$A_1(0,0,1)$

$B_1(1,0,1)$

$C_1(1,1,1)$

$D_1(0,1,1)$



a) $\vec{AB_1} = (1,0,1)$

$\vec{A_1C_1} = (1,1,0)$

$\cos\varphi = \dfrac{\vec{AB_1} \cdot \vec{A_1C_1}}{|\vec{AB_1}|\,|\vec{A_1C_1}|}$

$\cos\varphi = \dfrac{(1,0,1)\cdot(1,1,0)}{\sqrt{1+1}\cdot\sqrt{1+1}} = \frac{1}{2}$

$\varphi = 60° = \frac{\pi}{3}$

b) $\vec{AB_1} \cdot \vec{A_1C_1} = (1,0,1) \cdot (1,1,0) = 1$

c) $E = \left(\frac{1+0}{2}, \frac{1+0}{2}, \frac{0+0}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$

$\vec{B_1T} : \vec{TE} = 2:1$

$$\vec{r_T} = \frac{\vec{r_{B_1}} + 2\vec{r_E}}{1+2}$$

$$\vec{r_T} = \frac{(1,0,1)+(1,1,0)}{3}$$

$$\vec{r_T} = \frac{(2,1,1)}{3}$$

$$T = \left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

6. Analitička geometrija u ravni

6.1. Tačka i prava

- Svaka tačka T u ravni se zadaje svojim koordinatama $T(x_1, y_1)$
- Vektor koji spaja tačke $A(x_1, y_1)$ i $B(x_2, y_2)$ je:

$\vec{AB} = (x_2 - x_1, y_2 - y_1)$

- Rastojanje tačaka A i B je:

$d = AB = |\vec{AB}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

- Središte duži AB je $S\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
- Jednačina prave:

Opšti oblik: $y = kx + n$

$k = tg\alpha$ - koeficijent pravca

n - odsečak na y-osi

- Specijalni slučajevi pravih:

Jednačina prave normalne na x-osu $x = m$

Jednačina prave normalne na y-osu $y = n$

- Prava kroz tačku $A(x_1, y_1)$ sa koeficijentom pravca k:

$y - y_1 = k(x - x_1)$